The Nielsen numbers of iterations of maps on infra-solvmanifolds of type \((\mathbb{R})\) and periodic orbits

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I would like to thank the Organizers, especially Daciberg Goncalves, for invitation!
This talk is based on the manuscript

A. Fel’shtyn and J. B. Lee,

The Nielsen numbers of iterations of maps on infra-solvmanifolds of type (R) and periodic points, arXiv:1403.7631.
For self-maps $f$ on infra-solvmanifolds of type $(R)$, we have studied

1. Nielsen numbers $N(f^k)$
2. Nielsen zeta function $N_f(z)$
3. Asymptotic behavior of the sequence $\{N(f^k)\}$
4. Essential periodic orbits
5. Homotopy minimal periods

In this talk, I will concentrate on “Essential periodic orbits and Homotopy minimal periods”.
A study of the Lefschetz numbers of the iterates of a map is carried to obtain conditions for the existence of periodic points.

Our goal in this article is:

When \( f : M \rightarrow M \) be a continuous map on an infra-solvmanifold \( M = \Pi \backslash S \) of type \((R)\),

utilizing the arguments employed mainly in [2] and [3, Chap. III] for the Lefschetz numbers of iterations,

we study the asymptotic behavior of the sequence of the Nielsen numbers \( \{ N(f^k) \} \), the essential periodic orbits of \( f \) and the homotopy minimal periods of \( f \) by using the Nielsen theory of maps \( f \) on infra-solvmanifolds of type \((R)\).
Let $S$ be a connected and simply connected solvable Lie group. Consider $\text{Aff}(S) = S \rtimes \text{Aut}(S)$, the affine group of $S$. Let $\Pi \subset \text{Aff}(S)$ be a torsion-free finite extension of the lattice $\Gamma := \Pi \cap S$ of $S$. Hence $\Pi$ fits the following commutative diagram of short exact sequences

\[
\begin{array}{cccccc}
1 & \longrightarrow & S & \longrightarrow & \text{Aff}(S) & \longrightarrow & \text{Aut}(S) & \longrightarrow & 1 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & \Phi & \longrightarrow & 1
\end{array}
\]

where $\Phi = \Pi / \Gamma$ is a finite group, called the holonomy group of $\Pi$. It sits naturally in $\text{Aut}(S)$.
Infra-solvmanifolds of type \((\mathbb{R})\)

In this case, \(\Pi\) acts freely on \(S\) and the manifold \(\Pi \backslash S\) is called an infra-solvmanifold. Thus every infra-solvmanifold \(\Pi \backslash S\) is finitely covered by the special solvmanifold \(\Gamma \backslash S\).

When \(S\) is of type \((\mathbb{R})\), i.e., when \(\text{ad}X : \mathcal{G} \rightarrow \mathcal{G}\) has only real eigenvalues for all \(X \in \mathcal{G}\), we say that the manifold \(\Pi \backslash S\) is of type \((\mathbb{R})\).

Useful facts are:
When \(S\) is of type \((\mathbb{R})\),

1. every endomorphism on \(\Gamma\) extends uniquely to a Lie group endomorphism on \(S\),
2. \(\exp : \mathcal{G} \rightarrow S\) is a diffeomorphism.
Infra-solvmanifolds of type $(\mathbb{R})$: example

Take $S = \mathbb{R}^n$
Then $\text{Aff}(S) = S \rtimes \text{Aut}(S) = \mathbb{R}^n \rtimes \text{GL}(n, \mathbb{R})$.

In this case, the infra-solvmanifolds $\Pi \backslash S = \Pi \backslash \mathbb{R}^n$ are flat manifolds which are finitely covered by the torus $\Gamma \backslash S = \mathbb{Z}^n \backslash \mathbb{R}^n$.

When $S$ is a nilpotent Lie group, we obtain infra-nilmanifolds.
Averaging formula

Let $f : M \to M$ be a continuous map on an infra-solvmanifold $M = \Pi \backslash S$ of type $(\mathbb{R})$ with holonomy group $\Phi$.

Then $f$ admits an affine homotopy lift $(d, D)$.
That is, $\exists d \in S$ and $D : S \to S$, a Lie group endomorphism, such that $\tilde{f} \simeq (d, D) : S \to S$.

Then

$$N(f^k) = \frac{1}{\# \Phi} \sum_{A \in \Phi \subset \text{Aut}(S)} |\det(I - A_*D_*^k)|,$$

where $A_*$ and $D_*$ are differentials of $A$ and $D$. 
Necessary facts

Nielsen zeta function

Let $f : M \to M$ be a continuous map on an infra-solvmanifold $M = \Pi \backslash S$ of type $(\mathbb{R})$.

Then the Nielsen zeta function

$$N_f(z) = \exp \left( \sum_{k=1}^{\infty} \frac{N(f^k)}{k} z^k \right)$$

is a rational function.

Gauss congruence

Let $f: M \to M$ be a continuous map on an infra-solvmanifold $M = \Pi \backslash S$ of type (R).

Then the following Gauss congruences hold:

$$\sum_{d|k} \mu\left(\frac{k}{d}\right) N(f^d) \equiv 0 \mod k$$

for all $k > 0$.

A. Fel’shtyn and J. B. Lee,
$N_f(z)$ is rational
+ Gauss congruences for Nielsen numbers

$$\sum_{d|k} \mu \left( \frac{k}{d} \right) N(f^d) \equiv 0 \mod k$$

(on infra-solvmanifolds of type (R))

$\Rightarrow \exists$ a compact polyhedron $X$ and $g : X \to X$ so that

$$N(f^k) = L(g^k)$$

for all $k > 0$ by Dold [1].
Consequences

\[ N(f^k) = L(g^k) = \text{tr}(M^k) \] for some integer matrix \( M \) by [2] + Zarelua [1]:

\[
\sum_{d|k} \mu\left(\frac{k}{d}\right) N(f^d) \equiv 0 \pmod{k},
\]

\[ N(f^{p^r}) \equiv N(f^{p^r-1}) \pmod{p^r}. \]

A. V. Zarelua,
On congruences for the traces of powers of some matrices Tr.
Mat. Inst. Steklova 263 (2008), 85–105 (Russian); translation in
Algebraic Multiplicities

From Gauss congruences

\[ \sum_{d \mid k} \mu \left( \frac{k}{d} \right) N\left( f^d \right) \equiv 0 \mod k, \]

consider the algebraic multiplicities

\[ A_k(f) := \frac{1}{k} \sum_{d \mid k} \mu \left( \frac{k}{d} \right) N\left( f^d \right) \in \mathbb{Z}. \]

By Mobius inversion,

\[ N(f^k) = \sum_{d \mid k} d \cdot A_d(f). \]
Nielsen numbers $N(f^k)$

From the rationality of $N_f(z)$,

$$N_f(z) = \exp \left( \sum_{k=1}^{\infty} \frac{N(f^k)}{k} z^k \right) = \prod_{i=1}^{r(f)} (1 - \lambda_i z)^{-\rho_i}$$

Taking log on both sides, we have

$$\sum_{k=1}^{\infty} \frac{N(f^k)}{k} z^k = \sum_{i=1}^{r(f)} -\rho_i \log(1 - \lambda_i z) = \sum_{i=1}^{r(f)} \rho_i \left( \sum_{k=1}^{\infty} \frac{(\lambda_i z)^k}{k} \right)$$

$$= \sum_{k=1}^{\infty} \frac{r(f) \sum_{i=1}^{\infty} \rho_i \lambda_i^k}{k} z^k$$

Hence

$$N(f^k) = \sum_{i=1}^{r(f)} \rho_i \lambda_i^k.$$
Asymptotic Behavior

Define
\[ \lambda(f) = \max\{|\lambda_i| : i = 1, \cdots, r(f)\}. \]

Then we can write
\[ \frac{N(f^k)}{\lambda(f)^k} = \frac{\Gamma(f^k)}{\lambda(f)^k} + \frac{\Omega(f^k)}{\lambda(f)^k}, \]
where
\[ \Gamma(f^k) = \sum_{|\lambda_j| = \lambda(f)} \rho_j \lambda_j^k, \quad \Omega(f^k) = \sum_{|\lambda_i| < \lambda(f)} \rho_i \lambda_i^k. \]

Hence asymptotically
\[ \frac{N(f^k)}{\lambda(f)^k} = \frac{\Gamma(f^k)}{\lambda(f)^k} = \sum \rho_j e^{2i\pi(k\theta_j)}. \]

This is periodic when all \( \theta_j \) are rationals.
Corollary

If \( \lambda(f) \geq 1 \), then

\[
\limsup N(f^k) / \lambda(f)^k = \limsup \Gamma(f^k) / \lambda(f)^k > 0.
\]

Corollary

If \( \lambda(f) > 1 \), then there exists \( N \) such that

\[
\forall m \geq N, \quad \exists \ell \in \{0, 1, \cdots, n(f) - 1\} \text{ such that } A_{m+\ell}(f) \neq 0.
\]
Lefschetz gives periodic points (orbits)
Nielsen gives essential periodic points (orbits)

We shall give an estimate from below the number of essential periodic orbits of maps on infra-solvmanifolds of type (R).

Let

$$\mathcal{O}(f, k) = \{ \langle F \rangle \mid F \text{ is a essential fixed point class of } f^m \text{ with } m \leq k \}.$$ 

That is, $\mathcal{O}(f, k)$ is the set of all essential periodic orbits of $f$ with length $\leq k$. 
A known result

Recall

**Theorem (Shub-Sullivan, 1974 Topology)**

\[ f: M \rightarrow M \text{ is a } C^1\text{-map on a smooth compact manifold } M \text{ and} \]
\[ \{L(f^k)\} \text{ is unbounded, then the set of periodic points of } f, \]
\[ \bigcup_k \text{Fix}(f^k), \text{ is infinite.} \]

This theorem is not true for continuous maps. Consider the one-point compactification of the map of the complex plane \( f(z) = 2z^2/|z| \).
This is a continuous degree two map of \( S^2 \) with only two periodic points. But \( L(f^k) = 2^{k+1} \).

However, for any continuous maps \( f \) on an infra-solvmanifold of type (R), by the averaging formula, we obtain \( |L(f^k)| \leq N(f^k) \). If \( L(f^k) \) is unbounded, then so is \( N(f^k) \) and hence the number of essential fixed point classes of all \( f^k \) is infinite.
Now about essential periodic orbits, we have

**Theorem**

Let $f$ be a map on an infra-solvmanifold of type $(R)$. Suppose that the sequence $N(f^k)$ is unbounded. Then there exists a natural number $N_0$ such that

$$k \geq N_0 \implies \#O(f, k) \geq \frac{k - N_0}{r(f)}.$$ 

In particular, the set of essential periodic orbits of $f$ is infinite.
Will study (homotopy) minimal periods of maps $f$ on infra-solvmanifolds of type $(\mathbb{R})$. We like to determine $\text{HPer}(f)$ only from the knowledge of the sequence $\{N(f^k)\}$.

Fix$(f) = \{x \in X \mid f(x) = x\}$,

Per$(f) =$ the set of all minimal periods of $f$,

$P_n(f) =$ the set of all periodic points of $f$ with minimal period $n$,

$\text{HPer}(f) = \bigcap_{g \simeq f} \{n \in \mathbb{N} \mid P_n(g) \neq \emptyset\}$

$= \text{the set of all homotopy minimal periods of } f$. 
For maps on tori, Alsedà-Baldwin-Llibre-Swanson-Szlenk; Halpern; Jiang-Llibre; and so on.

For maps on nilmanifolds and some solvmanifolds (e.g., special solvmanifolds modeled on $\text{Sol}$ and $\text{Sol}_1^4$), Jezierski-Kedra-Marzantowicz; Lee-Zhao; Ha-Lee; · · ·.

**Infra-homogeneous spaces**

For some maps on flat manifolds, Hoffman-Liang-Sakai-Zhao.

For hyperbolic maps on infra-nilmanifolds, Dekimpe-Dugardein.

For some maps on infra-solvmanifolds of type $(\mathbb{R})$, Lee-Zhao.
A self-map is **essentially reducible** if any fixed point class of $f^k$ being contained in an essential fixed point class of $f^{kn}$ is essential. A space is **essentially reducible** if every self-map is essentially reducible. Every infra-solvmanifold of type (R) is essentially reducible.

**Theorem (ABLSS)**

Let $f : M \to M$ be an essentially reducible map. If

$$
\sum_{\substack{m \text{ prime} \\ m \mid k}} N(f^k) < N(f^m),
$$

then any map which is homotopic to $f$ has a periodic point with minimal period $m$, i.e., $m \in \text{HPer}(f)$. 

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Theorem (Jezierski)

$f$ is homotopic to $g$ with $P_k(g) \neq \emptyset$ iff $NP_k(f) \neq 0$, where

$$NP_k(f) = \text{(number of irreducible essential orbits of Reidemeister classes of } f^k) \times k.$$ 

Hence

$$\text{HPer}(f) = \{k \mid \exists \text{ irreducible essential fixed point class of } f^k\}.$$ 

The lower density of the homotopy minimal periods of $f$:

$$DH(f) = \liminf_{n \to \infty} \frac{\#(\text{HPer}(f) \cap [0, n])}{n}.$$ 

(See JM’s book or Zhao’s paper.)
Theorem

Let $f$ be a map on an infra-solvmanifold of type $(R)$ with $\lambda(f) > 1$. Suppose that the sequence $N(f^k)/\lambda(f)^k$ is asymptotically periodic. Then there exist $m$ and an infinite sequence $\{p_i\}$ of primes such that $\{mp_i\} \subset \text{HPer}(f)$.

Corollary

Let $f$ be a map on an infra-solvmanifold of type $(R)$. Suppose that the sequence $N(f^k)$ is eventually strictly monotone increasing. For example $f$ is an expanding map. Then $\text{HPer}(f)$ is cofinite.
Recalling

\[ \mathcal{N}(f^k) = \sum_{i=1}^{r(f)} \rho_i \lambda_i^k \]

and

\[ \lambda := \lambda(f) = \max\{|\lambda_i| : i = 1, \cdots, r(f)\}. \]

we define

\[ \Gamma(f^k) = \sum_{|\lambda_i| = |\lambda|} \rho_i \lambda_i^k, \quad \tilde{\mathcal{N}}(f^k) = \frac{1}{|\lambda|^k} \Gamma(f^k) \]
**STEP 1** If $\lambda(f) \geq 1$, then

$$\limsup \frac{N(f^k)}{\lambda(f)^k} = \limsup |\tilde{N}(f^k)| > 0.$$  

**STEP 2** By assumption, the sequence $\{N(f^k)/\lambda(f)^k\}_k$ is asymptotically $q$-periodic and nonzero (by Step 1). Hence $\tilde{N}(f^m) \neq 0$ for some $m$ and $\tilde{N}(f^m) = \tilde{N}(f^{m+\ell q})$.

Let $h = f^m$. Then $\lambda(h) = \lambda(f^m) = \lambda(f)^m$. The periodicity for $f$ induces $\tilde{N}(h^{1+\ell q}) = \tilde{N}(h) > 0$ for some $q$ and all $\ell > 0$. 

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**STEP 3** If $\lambda(h) > 1$, then

$$N(h^m) > \epsilon > 0, \forall m > N \Rightarrow |A_m(h)| > 0.$$  

Hence $|A_{1+\ell q}(h)| > 0$ for $\ell$ sufficiently large. By Dirichlet prime theorem, there are infinitely many primes $p$ of the form $1 + \ell q$.

**STEP 4** If $|A_p(h)| > 0$, then $p$ is the minimal period of some essential periodic point of $h$.

Thus $mp$ is a period of $f$, and $m'p$ is the minimal period of $f$ for some $m' | m$.  

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Finally, we consider all primes $p_i$ of the form $1 + \ell q$ and all $m_i$ with $m_i \mid m$.
Choose a constant subsequence $\{m_{i_k}\}$ of $\{m_i\}$, say $\{m_0\}$. Then $m_0 p_{i_k}$ is a minimal period of $f$.

These arguments also work for all maps homotopic to $f$. Hence $m_0 p_{i_k} \in \text{HPer}(f)$. 