Collapsing of Eilenberg–Moore spectral sequences converging to certain based gauge groups

Younggi Choi
Department of Mathematics Education,
Seoul National University

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For a path connected space $X$ and the path–loop fibration,

$$\Omega X \to PX \to X$$

we consider the Eilenberg–Moore spectral sequence $\{E^r_*, d_r\}$ converging to $H^*(\Omega X; \mathbb{F}_p)$ with

$$E_2 \cong \text{Tor}_{H^*(X; \mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p).$$
Let $G$ be a compact simple Lie group.

$$\Omega G \to PG \to G$$

**Question:**
Does the Eilenberg–Moore spectral sequence of the path loop fibration converging to the mod $p$ (co)homology of the loop space of any compact simple Lie group collapse at the $E_2$–term?

This does not hold. In fact, the Eilenberg–Moore spectral sequences of the path loop fibrations $\Omega G \to PG \to G$ converging to the mod $p$ cohomology of the single loop spaces of compact simple Lie groups do not collapse at the $E_2$–term in the cases of exceptional Lie groups


**Theorem**[Bott, Lin]

The cohomology of the loop space of any simply connected finite $H$–space is concentrated on even degrees and of torsion free.
Theorem[Choi,Yoon]

The Eilenberg–Moore spectral sequences of the path loop fibrations converging to the mod $p$ (co)homology of the double and the triple loop spaces of any compact simple Lie group collapse at the $E^2$–term.

\[
\Omega^2 G \to P\Omega G \to \Omega G \\
\Omega^3 G \to P\Omega^2 G \to \Omega^2 G
\]

Theorem[Lin]

The Eilenberg–Moore spectral sequences of the path loop fibrations converging to the mod $p$ (co)homology of the double and the triple loop spaces of any simply connected finite $H$–space collapse at the $E^2$–term.

\[
\Omega^2 H \to P\Omega H \to \Omega H \\
\Omega^3 H \to P\Omega^2 H \to \Omega^2 H
\]
\[ \Omega^4_0 G \rightarrow P\Omega^3_0 G \rightarrow \Omega^3_0 G \]

**Question:**
Does the Eilenberg–Moore spectral sequences of the path loop fibrations converging to the mod $p$ cohomology of the four fold loop spaces of any compact simple Lie group collapse at the $E_2$–term?

Yes, this holds for any compact simple Lie groups
gauge group of bundle automorphisms

\[ G : \text{Compact, connected simple Lie group} \]
\[ \longrightarrow \]
\[ G \]

\[ BG : \text{Classifying space for } G \]
\[ \longrightarrow \]
\[ BG \]

\[ P_k : \text{Principal bundle over } S^4 \text{ classified by} \]
\[ \text{the map} S^4 \rightarrow BG \text{ of degree } k \in \mathbb{Z} \]
\[ \longrightarrow \]
\[ S^4 \]
\[ \longrightarrow \]
\[ k \]
\[ BG \]

\[ \mathcal{G}_k(G) : \text{The (full) gauge group of bundle automorphisms on } P_k, \]
\[ \text{that is, } G\text{–equivariant self maps of } P_k \text{ covering identity map of } S^4. \]

\[ \mathcal{G}_k^b(G) : \text{The based gauge group which consists of base point} \]
\[ \text{preserving automorphisms on } P_k, \text{ that is, } G\text{–equivariant self maps} \]
\[ \text{of } P_k \text{ covering identity map of } S^4 \text{ which fix one fiber.} \]
The gauge group $G_k(G)$ acts freely on $\text{Map}(P_k, EG)$ and its orbit space is $\text{Map}_k(S^4, BG)$.

The gauge group $G^b_k(G)$ acts freely on $\text{Map}_*(P_k, EG)$ and its orbit space is $\text{Map}_*(S^4, BG) = \Omega^3_k G$.

That is, we have

$$BG_k(G) \cong \text{Map}_k(S^4, BG), \quad BG^b_k(G) \cong \Omega^3_k G.$$
The gauge group:

\[ \Omega^3 G \to \Map_{p_k}(S^4, BG) \cong BG_k(G) \to BG \]

\[ \Omega(\Omega^3_k G) \to \Omega BG_k(G) \to \Omega BG \]

\[ \Omega BX \cong X, B\Omega X \cong X_0 \]

So we get

\[ \Omega(\Omega^3_k G) = \Omega^4 G \to \Omega BG_k(G) \to \Omega BG \]

\[ \Omega^4 G \to G_k(G) \to G \]
Theorem
For a path connected $H$-space, the following statements are equivalent.
(a) The Eilenberg–Moore spectral sequence for the path-loop fibration $\Omega X \to PX \to X$ converging to $H^*(\Omega X; Z_p)$ collapses at the $E_2$–term.
(b) $\ker \sigma^* = 0$. 
We denote the primitives and the indecomposables of $H^*(X; \mathbb{F}_p)$ by $P^*(X; \mathbb{F}_p)$ and $Q^*(X; \mathbb{F}_p)$, respectively. In the Eilenberg–Moore spectral sequence converging to $H^*(\Omega X; \mathbb{F}_p)$ for the path–loop fibration, we have

$$\sigma^*: Q^*(X; \mathbb{F}_p) \cong Tor^{-1,*}_{H^*(X; \mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p) = E_2^{-1,*} \to E_\infty^{-1,*} \subset H^*(\Omega X; \mathbb{F}_p).$$

Since the elements of $Tor^{-1,*}_{H^*(X; \mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p)$ are primitive and permanent cycles in the Eilenberg–Moore spectral sequence, the above map induces the suspension homomorphism

$$\sigma^*: Q^*(X; \mathbb{F}_p) \to P^*(\Omega X; \mathbb{F}_p).$$
The homology suspension map

\[ \sigma_* : QH_*(\Omega X; \mathbb{F}_p) \to PH_{*+1}(X; \mathbb{F}_p). \]

**Theorem[Browder]**

Let \( X \) be a path connected, simply connected \( H \)-space. Then the following are true.

(a) The suspension \( \sigma_* : QH_k(\Omega X; \mathbb{F}_p) \to PH_{k+1}(X; \mathbb{F}_p) \) is injective if \( k \neq p^t(2n) - 2, \ t \geq 1 \) where \( 2n \) is the dimension of some generator of \( H^*(X; \mathbb{F}_p) \).

(b) The suspension \( \sigma_* : QH_k(\Omega X; \mathbb{F}_p) \to PH_{k+1}(X; \mathbb{F}_p) \) is surjective if \( k \neq p^t(2m) + 1, \ t \geq 1 \) where \( 2m \) is the dimension of some generator of \( H^*(\Omega X; \mathbb{F}_p) \).

(c) The elements in \( \ker \sigma_* \) are dual to elements in the image of the cohomology transpotence.
The End

Thank you!