AUTOMORPHISMS OF UNIFORM LATTICES OF NILPOTENT LIE GROUPS UP TO DIMENSION FOUR

JONG BUM LEE AND SANG RAE LEE

Abstract. In this paper, when $G$ is a connected and simply connected nilpotent Lie group of dimension less than or equal to four, we study the uniform lattices $\Gamma$ of $G$ up to isomorphism and then we study the structure of the automorphism group $\text{Aut}(\Gamma)$ of $\Gamma$ from the viewpoint of splitting as a natural extension.

1. Introduction

In this paper we study the group of automorphisms of any uniform lattice of a connected and simply connected nilpotent Lie group $G$ up to dimension four. This work was motivated by the papers [3] and [5], in which the authors considered the discrete subgroup $\text{Heis}(3, \mathbb{Z})$ of the three-dimensional Heisenberg group $\text{Heis}(3, \mathbb{R})$ and proved that the automorphism group $\text{Aut}(\text{Heis}(3, \mathbb{Z}))$ admits a splitting as a natural extension of $\mathbb{Z}^2$ by $\text{GL}(2, \mathbb{Z})$.

The connected and simply connected nilpotent Lie groups of dimension less than or equal to four are well understood. In dimension one or two, there is only one such Lie group, the abelian Lie group $\mathbb{R}$ or $\mathbb{R}^2$. There are two connected and simply connected three-dimensional nilpotent Lie groups $\mathbb{R}^3$ and $\text{Nil}^3$. The Lie group $\text{Nil}^3$ is the Heisenberg group

$$\text{Heis}(3, \mathbb{R}) = \left\{ \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$ 

There are three connected and simply connected four-dimensional nilpotent Lie groups $\mathbb{R}^4$, $\text{Nil}^3 \times \mathbb{R}$ and $\text{Nil}^4$. The Lie groups $\text{Nil}^3 \times \mathbb{R}$ and $\text{Nil}^4$ are of the form $\mathbb{R}^3 \rtimes \varphi(s) \mathbb{R}$, [4], where $\varphi(s)$ is respectively

$$\varphi(s) = \begin{pmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & s/2 & s^2 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}. $$

For solvable Lie groups, one may refer to [6, 7].

2010 Mathematics Subject Classification. Primary 22E25; Secondary 22E40.
Key words and phrases. automorphism, nilpotent Lie group, splitting, uniform lattice.

The first-named author is supported in part by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (NRF2016R1D1A1B01006971).
A discrete subgroup $\Gamma$ of a Lie group $G$ is called a uniform lattice of $G$ if its orbit space $\Gamma \backslash G$ is compact. When $G$ is an abelian Lie group $\mathbb{R}^n$, every uniform lattice $\Gamma$ of $G$ is isomorphic to $\mathbb{Z}^n \subset \mathbb{R}^n$ and hence $\text{Aut}(\Gamma) \cong \text{Aut}(\mathbb{Z}^n) \cong \text{GL}(n, \mathbb{Z})$.

In this paper, when $G$ is $\text{Nil}^3$, $\text{Nil}^3 \times \mathbb{R}$ or $\text{Nil}^4$, we first study the uniform lattices $\Gamma$ of $G$ up to isomorphism and then we study the structure of the automorphism group $\text{Aut}(\Gamma)$ of $\Gamma$ from the viewpoint of splitting as a natural extension.

2. The Lie group $\text{Nil}^3$

The Lie group $\text{Nil}^3$ is the Heisenberg group

$$\text{Heis}(3, \mathbb{R}) = \left\{ \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$  

We will study $\text{Aut}(\Gamma)$ for any uniform lattice $\Gamma$ of $\text{Nil}^3$. The groups

$$\Gamma_k = \left\{ \begin{pmatrix} 1 & n & \ell \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} \mid \ell, m, n \in \mathbb{Z} \right\}$$

are uniform lattices of $\text{Nil}^3$. Note that $\Gamma_1 = \text{Heis}(3, \mathbb{Z})$, and $\Gamma_{-k} \cong \Gamma_k$ for all $k \neq 0$. It is known that any uniform lattice of $\text{Nil}^3$ is isomorphic to exactly one $\Gamma_k$ for some $k > 0$. In [3] and [5], it is shown that the automorphism group of $\Gamma_1$ is isomorphic to the group $\mathbb{Z}^2 \rtimes \text{GL}(2, \mathbb{Z})$. Utilizing the methods employed in [3] and [5], we can prove that $\text{Aut}(\Gamma_k)$ is isomorphic to $\mathbb{Z}^2 \rtimes \text{GL}(2, \mathbb{Z})$ for every $k > 0$.

**Lemma 2.1.** The lattice $\Gamma_k$ of $\text{Nil}^3$ may be presented as

$$\Gamma_k = \langle \alpha, \beta, \gamma \mid [\alpha, \beta] = \gamma^{-k}, [\gamma, \alpha] = [\gamma, \beta] = 1 \rangle.$$  

with $\alpha, \beta$ and $\gamma$ corresponding to the generators

$$\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 & \frac{1}{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  

Because the center $\mathbb{Z}(\Gamma_k)$ of $\Gamma_k$ is generated by $\gamma$, every automorphism of $\Gamma_k$ induces an automorphism of the quotient group $\Gamma_k/\mathbb{Z}(\Gamma_k) \cong \mathbb{Z}^2$. Thus we have a natural homomorphism $\vartheta : \text{Aut}(\Gamma_k) \rightarrow \text{GL}(2, \mathbb{Z})$. Indeed it is well-known that $\vartheta$ is surjective, see for example [3, Proposition 6]. The purpose of this section is, using the Lie algebra argument, to prove that $\vartheta : \text{Aut}(\Gamma_k) \rightarrow \text{GL}(2, \mathbb{Z})$ splits.

Recall that the Lie algebra nil$^3$ of $\text{Nil}^3$ is

$$\text{nil}^3 = \left\{ \begin{pmatrix} 0 & b & c \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}.$$
Choose the canonical basis \( \{ e_1, e_2, e_3 \} \) in \( \mathfrak{nil}^3 \) where
\[
    e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
Then \( [e_1, e_2] = -e_3 \) and \( [e_3, e_1] = [e_3, e_2] = 0 \). With respect to the canonical basis, each automorphism of \( \mathfrak{nil}^3 \) has a matrix presentation.

**Proposition 2.2** ([2, Proposition 2.2]). The group \( \text{Aut}(\mathfrak{nil}^3) \) of all automorphisms of the Lie algebra \( \mathfrak{nil}^3 \) is isomorphic to the matrix group
\[
\left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ u & v & ad - bc \end{pmatrix} \mid a, b, c, d, u, v \in \mathbb{R}, \; ad - bc \neq 0 \right\}.
\]

Since the Lie group \( \text{Nil}^3 \) is simply connected, we have a canonical isomorphism \( \text{Aut}(\text{Nil}^3) \cong \text{Aut}(\mathfrak{nil}^3) \) defined by \( \phi \mapsto d\phi \) fitting in the following commuting diagram.

\[
\begin{array}{ccc}
\mathfrak{nil}^3 & \xrightarrow{d\varphi} & \mathfrak{nil}^3 \\
\log \uparrow & & \exp \downarrow \\
\text{Nil}^3 & \xrightarrow{\varphi} & \text{Nil}^3
\end{array}
\]

Using the diffeomorphism
\[
\exp : \mathfrak{nil} \to \text{Nil}, \quad \begin{pmatrix} 0 & b & c \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & b & c + \frac{1}{2}ab \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}
\]
we can see that if
\[
d\varphi = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ u & v & ad - bc \end{pmatrix}
\]
then
\[
(2.2) \quad \varphi = \exp \circ d\varphi \circ \log : \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & cx + dy & z^* \\ 0 & 1 & ax + by \\ 0 & 0 & 1 \end{pmatrix}
\]

where \( z^* = (ad - bc)z + \frac{1}{2}acx^2 + ux + bcxy + vy + \frac{1}{2}bdy^2 \). In what follows, we shall identify \( \phi = d\varphi \) with \( \varphi \).

Consider the uniform lattice \( \Gamma_k \) of \( \text{Nil}^3 \). Due to Mal’cev, every automorphism of \( \Gamma_k \) can be extended uniquely to a Lie group automorphism of \( \text{Nil}^3 \). This implies that we can regard \( \text{Aut}(\Gamma_k) \) as a subgroup of \( \text{Aut}(\text{Nil}^3) \). Thus
\[
\text{Aut}(\Gamma_k) \subset \text{Aut}(\mathfrak{nil}^3) = \text{Aut}(\mathfrak{nil}^3)
\]
as a subgroup of the matrix group GL(3, ℝ). Furthermore, we have a commutative diagram between surjective homomorphisms

\[
\begin{array}{ccc}
\text{Aut(nil}_3^3) & \overset{\tilde{\vartheta}}{\longrightarrow} & \text{Aut}(\mathbb{R}^2) = \text{GL}(2, \mathbb{R}) \\
\uparrow & & \uparrow
\end{array}
\]

\[
\begin{array}{ccc}
\text{Aut(Γ}_k^k) & \overset{\vartheta}{\longrightarrow} & \text{GL}(2, \mathbb{Z}) \\
\end{array}
\]

where the vertical maps are inclusions, and

\[
\tilde{\vartheta} : \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ u & v & ad - bc \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

Now we recall that GL(2, Z) is presented as

\[
\langle \rho, \tau, \kappa \mid \rho\tau\rho = \tau\rho\tau, (\rho\tau\rho)^4 = 1, \kappa\rho\kappa^{-1} = \rho^{-1}, \kappa\tau\kappa^{-1} = \tau^{-1}, \kappa^2 = 1 \rangle,
\]

where \(\rho, \tau\) and \(\kappa\) may be taken to correspond to the matrices

\[
\rho = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tau = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \kappa = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

We seek an explicit section \(\eta\) of \(\tilde{\vartheta}\). Consider the following elements of \(\text{Aut(nil}_3^3)\):

\[
R = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ u_1 & v_1 & 1 \end{pmatrix}, T = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ u_2 & v_2 & 1 \end{pmatrix}, K = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ u_3 & v_3 & -1 \end{pmatrix}
\]

As observed above, these elements are regarded as elements of \(\text{Aut(Nil}_3^3)\). Assume that the elements \(R, T, K\) satisfy the defining relations of \(\text{GL}(2, \mathbb{Z})\).

\[
RT R = T R T, \quad (R T R)^4 = I_3, \quad (R T R)^2 \neq I_3,
\]

\[
K^2 = I_3, \quad K R K^{-1} = R^{-1}, \quad K T K^{-1} = T^{-1}.
\]

By direct computation with (2.2) we have

\[
R(\alpha) = \alpha \gamma^{u_1 k}, \quad R(\beta) = \alpha \beta \gamma^{(v_1 + 1/2) k}, \quad R(\gamma) = \gamma,
\]

\[
T(\alpha) = \alpha \beta^{-1} \gamma^{(v_2 - 1/2) k}, \quad T(\beta) = \beta \gamma^{v_2 k}, \quad T(\gamma) = \gamma,
\]

\[
K(\alpha) = \alpha^{-1} \gamma^{v_3 k}, \quad K(\beta) = \beta \gamma^{v_3 k}, \quad K(\gamma) = \gamma^{-1}.
\]

Using \([\alpha, \gamma] = [\beta, \gamma] = 1\), it is immediate to check the implications

\[
R T R(\alpha) = T R T(\alpha) \Rightarrow v_2 = 0,
\]

\[
K^2(\alpha) = \alpha \Rightarrow u_3 = 0,
\]

\[
K R K^{-1}(\alpha) = R^{-1}(\alpha) \Rightarrow u_1 = 0,
\]

\[
K T K^{-1}(\alpha) = T^{-1}(\alpha) \Rightarrow 2 u_2 + v_3 = -1.
\]
We can choose $v_1, u_2, v_3$ appropriately so that $R, T, K$ also preserve $\Gamma_k$. For example, one can take

$$R = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}, \quad K = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & -1 \end{pmatrix}$$

and define $\eta : \text{GL}(2, \mathbb{Z}) \to \text{Aut}(\text{nil}^3)$ by $\rho \mapsto R$, $\tau \mapsto T$, $\kappa \mapsto K$. It is straightforward to check that the above $R, T$ and $K$ satisfy the remaining relations of $\text{GL}(2, \mathbb{Z})$. Therefore, the subgroup of $\text{Aut}(\Gamma_k) \subset \text{Aut}(\text{nil}^3)$ generated by $R, T$ and $K$ is isomorphic to $\text{GL}(2, \mathbb{Z})$. Consequently, $\eta$ provides a desired splitting of $\vartheta$. The splitting $\eta$ is independent of $\Gamma_k$.

**Remark 2.3.** The group structure of $\text{Aut}(\Gamma_k)$ is complicated. By considering the Lie algebra, we can embed $\text{Aut}(\Gamma_k)$ as a subgroup of the matrix group $\text{GL}(3, \mathbb{R})$. This makes easy to check whether a splitting function of the surjective homomorphism $\vartheta : \text{Aut}(\Gamma_k) \to \text{GL}(2, \mathbb{Z})$ is a “homomorphism”.

**Remark 2.4 (Automorphisms of $\Gamma_k$).** We have shown that the homomorphism

$$\vartheta : \text{Aut}(\Gamma_k) \subset \text{Aut}(\text{nil}^3) \subset \text{GL}(3, \mathbb{R}) \to \text{GL}(2, \mathbb{Z})$$

is surjective and splits. This in particular implies that

$$\text{Aut}(\Gamma_k) = \left\{ H = \begin{pmatrix} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & 0 \\ p_1 & p_2 & p_3 \end{pmatrix} \in \text{Aut}(\text{nil}^3) \mid H \text{ preserves } \Gamma_k \right\}.$$

It is now straightforward to observe further that

$$\text{Aut}(\Gamma_k) = \left\{ \begin{pmatrix} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & 0 \\ m_{11}m_{22} - m_{21}m_{12} \end{pmatrix} \in \text{Aut}(\text{nil}^3) \mid m_{ij} \in \mathbb{Z}, \quad p_i + \frac{1}{2}m_{11}m_{22} \in \mathbb{Z} \right\}.$$

Hence, the groups $\text{Aut}(\Gamma_k)$ are identical for all $k \neq 0$.

In conclusion we have that

$$\ker(\vartheta) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p_1 & p_2 & 1 \end{pmatrix} \mid p_1, p_2 \in \mathbb{Z} \right\} \cong \mathbb{Z} \oplus \mathbb{Z},$$

and

$$\text{Aut}(\Gamma_k) \cong (\mathbb{Z} \oplus \mathbb{Z}) \rtimes \text{GL}(2, \mathbb{Z})$$

where the $\text{GL}(2, \mathbb{Z})$-action on $\ker(\vartheta)$ is

$$g \cdot \mathbf{p} = \begin{pmatrix} p_1 & p_2 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1}.$$
3. The nilpotent Lie group \( \text{Nil}^3 \times \mathbb{R} \)

There are three simply connected four-dimensional nilpotent Lie groups: \( \mathbb{R}^4, \text{Nil}^3 \times \mathbb{R} \) and \( \text{Nil}^4 \). The Lie groups \( \text{Nil}^3 \times \mathbb{R} \) and \( \text{Nil}^4 \) are of the form \( \mathbb{R}^3 \rtimes \varphi(s) \mathbb{R} \), [4], where \( \varphi(s) \) is respectively

\[
\varphi(s) = \begin{pmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & s/2 & s^2 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}.
\]

The group law of \( \mathbb{R}^3 \rtimes \varphi(s) \mathbb{R} \) is

\[
(x, s)(y, t) = (x + \varphi(s)y, s + t),
\]

and it can be embedded affinely in \( \text{GL}(4, \mathbb{R}) \) as

\[
\left\{ \begin{pmatrix} \varphi(s) & x \\ 0 & 1 \end{pmatrix} \right\} \subset \text{GL}(4, \mathbb{R})
\]

where \( \varphi(s) \in \text{GL}(3, \mathbb{R}) \) and \( x \in \mathbb{R}^3 \) is a column vector. Remark that \( \text{Nil}^3 \times \mathbb{R} \) is 2-step and \( \text{Nil}^4 \) is 3-step. We will continue to study \( \text{Aut}(\Gamma) \) for a uniform lattice \( \Gamma \) of \( \text{Nil}^3 \times \mathbb{R} \) in this section and \( \text{Nil}^4 \) in the next section.

It is easy to see that any uniform lattice \( \Gamma \) of \( \text{Nil}^3 \times \mathbb{R} \) is isomorphic to the product \( \Gamma_k \times \mathbb{Z} \), where \( \Gamma_k \) is a uniform lattice of \( \text{Nil}^3 \), see for example [1, Corollary 6.2.5]. Hence \( \Gamma \) can be presented as

\[
\langle \alpha, \beta, \gamma, \delta \mid [\alpha, \beta] = \gamma^{-k}, [\gamma, \alpha] = [\gamma, \beta] = [\delta, \alpha] = [\delta, \beta] = [\delta, \gamma] = 1 \rangle
\]

with \( \alpha, \beta, \gamma \) and \( \delta \) corresponding to the generators

\[
\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]

\[
\gamma = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \delta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

Let us denote \( \Gamma_k \times \mathbb{Z} \) by \( \Gamma_{k,0} \). Since \( \mathbb{Z}(\Gamma_{k,0}) = \langle \gamma, \delta \rangle \), we obtain a canonical surjective homomorphism \( \vartheta : \text{Aut}(\Gamma_{k,0}) \to \text{GL}(2, \mathbb{Z}) \). We will show that \( \vartheta \) also splits. It suffices to show that \( \text{Aut}(\Gamma_k) \) can be regarded as a subgroup of \( \text{Aut}(\Gamma_{k,0}) \) so that the following diagram is commutative

\[
\begin{array}{ccccc}
\text{Aut}(\Gamma_{k,0}) & \xrightarrow{\vartheta} & \text{GL}(2, \mathbb{Z}) & \xrightarrow{1} & \\
\uparrow & & \uparrow & & \uparrow \\
\text{Aut}(\Gamma_k) & \xrightarrow{\vartheta} & \text{GL}(2, \mathbb{Z}) & \xrightarrow{1} & \\
\end{array}
\]

(3.1)
Every element of $\Gamma_{k,0}$ can be written as $\alpha^m \beta^n \gamma^p \delta^q$. It can be seen easily that

$$ (\alpha^m \beta^n \gamma^p \delta^q)^r = \alpha^{rm} \beta^{rn} \gamma^{rp} \delta^{rq} \cdot \beta^n \alpha^m = \alpha^m \beta^n \gamma^{knmn} $$

for all $r \in \mathbb{Z}$. For any $h \in \text{Aut}(\Gamma_{k,0})$, since $\mathcal{Z}(\Gamma_{k,0}) = \langle \gamma, \delta \rangle$, we must have

$$ h(\alpha) = \alpha^{m_{11}} \beta^{m_{21}} \gamma^{p_{21}} \delta^{p_{21}}, $$
$$ h(\beta) = \alpha^{m_{12}} \beta^{m_{22}} \gamma^{p_{12}} \delta^{p_{22}}, $$
$$ h(\gamma) = \gamma^e \delta^{e'}, \quad h(\delta) = \gamma^e \delta^{e''}, $$

for some integers $m_{ij}, p_{ij}$ and $e, e', e''$. Since $h$ preserves the commutator relations

$$ [\alpha, \beta] = \gamma^{-k}, \quad [\gamma, \alpha] = [\gamma, \beta] = [\delta, \alpha] = [\delta, \beta] = [\delta, \gamma] = 1, $$

it follows from (3.2) and (3.3) that $e' = 0$ and $e = m_{11}m_{22} - m_{12}m_{21}$. Notice that

1. $\bar{\vartheta}(h) = \left( \begin{array}{cc} m_{11} & m_{12} \\ m_{21} & m_{22} \end{array} \right)$,

2. any automorphism $h$ of $\Gamma_k$ can be regarded as an automorphism $h$ of $\Gamma_{k,0}$ by taking $p_{21} = p_{22} = e' = e_1 = 0, e_2 = 1$.

Consequently, we have shown that the diagram (3.1) is commutative and since $\vartheta$ splits by Section 2 it follows that $\bar{\vartheta}$ splits.

4. The nilpotent Lie group $\text{Nil}^4$

The nilpotent Lie group $\text{Nil}^4$ is the matrix group

$$ \text{Nil}^4 = \left\{ \begin{pmatrix} 1 & s & s^2 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z, s \in \mathbb{R} \right\} $$

and so its Lie algebra is

$$ \text{nil}^4 = \left\{ \begin{pmatrix} 0 & s & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c, s \in \mathbb{R} \right\}. $$

The commutator subgroup $[\text{Nil}^4, \text{Nil}^4]$ is $\mathbb{R}^2$ (with $s = z = 0$), and the center $\mathcal{Z}(\text{Nil}^4)$ of $\text{Nil}^4$ is $\mathbb{R}$ (with $s = y = z = 0$). Since $\text{Nil}^4 / \mathcal{Z}(\text{Nil}^4)$ is a three-dimensional nilpotent Lie group, it follows that it is isomorphic to $\text{Nil}^3$. That
is,

\[
\begin{array}{cclll}
\Nil^3 & \xrightarrow{\pi} & \mathcal{Z}(\Nil^4) & \longrightarrow & \Nil^4 \\
1 & \longrightarrow & \Nil^4 & \longrightarrow & \Nil^4/\mathcal{Z}(\Nil^4) & \longrightarrow & 1
\end{array}
\]

where the explicit isomorphism is given by

\[
\pi : \begin{pmatrix} 1 & s & \frac{1}{2}s^2 & * \\ 0 & 1 & s & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & s & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}.
\]

We can see further that \([\Nil^4, \Nil^4]/\mathcal{Z}(\Nil^4)\) is isomorphic to \(\mathcal{Z}(\Nil^3)\). Consequently, we obtain the following commutative diagram

\[
\begin{array}{ccccccc}
1 & \longrightarrow & \mathcal{Z}(\Nil^3) & \longrightarrow & \Nil^3 & \longrightarrow & \Nil^3/\mathcal{Z}(\Nil^3) & \longrightarrow & 1 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
1 & \longrightarrow & [\Nil^4, \Nil^4] & \longrightarrow & \Nil^4 & \longrightarrow & \Nil^4/[\Nil^4, \Nil^4] & \longrightarrow & 1 \\
\uparrow & & \uparrow & & \pi & & \uparrow & & \equiv \\
\mathcal{Z}(\Nil^4) & \longrightarrow & \mathcal{Z}(\Nil^4) & \longrightarrow & \Nil^4/\mathcal{Z}(\Nil^4) & \longrightarrow & 1 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
1 & & 1 & & 1 & & 1
\end{array}
\]

### 4.1. The lattices of \(\Nil^4\)

Noting that the subset of elements of \(\Nil^4\) with \(s, x, y, z \in \mathbb{Z}\) is not a group, we shall describe the uniform lattices of \(\Nil^4\).

Let \(\Gamma\) be a uniform lattice of \(\Nil^3\). Then \(\mathcal{Z}(\Nil^4) \cap \Gamma\) is a uniform lattice of \(\mathcal{Z}(\Nil^3) = \mathbb{R}\), and \(\Gamma/(\mathcal{Z}(\Nil^3) \cap \Gamma)\) is a uniform lattice of \(\Nil^4/\mathcal{Z}(\Nil^3) = \Nil^3\).
Hence $\Gamma$ fits in the following short exact sequences

\[
\begin{array}{cccccc}
1 & 1 \\
\uparrow & \uparrow \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \Gamma & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & 1 \\
\uparrow & \uparrow & \uparrow & \uparrow \\
\end{array}
\]

(4.1)

\[
\begin{array}{cccccc}
1 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \Gamma & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & 1 \\
\uparrow & \uparrow & \uparrow & \uparrow \\
\end{array}
\]

Since $\bar{\Gamma}$ is a uniform lattice of $\text{Nil}_4^3$, it is isomorphic to some $\Gamma_k$. Thus we can choose a set of generators $\alpha, \beta, \gamma, \delta$ of $\Gamma$ so that $\delta \in \mathcal{Z}(\text{Nil}_4^4)$, and $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ generate $\bar{\Gamma}$. We can assume that $[\bar{\alpha}, \bar{\beta}] = \bar{\gamma}^{-k}$ with $k > 0$. This implies that $\alpha, \beta, \gamma$ and $\delta$ may be taken to correspond to matrices in $\text{Nil}_4^4$ of the form

\[
\alpha = \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 1 & \frac{1}{m} & b \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]

\[
\gamma = \begin{pmatrix} 1 & 0 & 0 & c \\ 0 & 1 & 0 & \frac{1}{n} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \delta = \begin{pmatrix} 1 & 0 & 0 & d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

A direct computation shows that the commutators among $\alpha, \beta, \gamma, \delta$ are independent of $a$ and $b$, hence we may choose $a = b = 0$. Moreover, $[\alpha, \gamma] = 1$. Therefore, $\Gamma$ can be presented as

\[
\langle \alpha, \beta, \gamma, \delta \mid [\alpha, \beta] = \gamma^{-k}\delta^m, [\alpha, \gamma] = 1, \\
[\beta, \gamma] = \delta^n, [\alpha, \delta] = [\beta, \delta] = [\gamma, \delta] = 1 \rangle.
\]

(4.3)

We choose $a, b, c$ and $d > 0$ so that they satisfy

\[
a = b = 0, \quad dn = \frac{1}{k}, \quad kc = dm + \frac{1}{2}.
\]

(4.4)

Then we can assume that $k, n > 0$ and the choice (4.2) of matrices for $\alpha, \beta, \gamma$ and $\delta$ with the conditions (4.4) gives rise to a realization of an abstract group with presentation (4.3) as a uniform lattice of $\text{Nil}_4^4$. We shall denote this $\Gamma$ by
\[ \Gamma_{k,m,n} \text{ with } k, n \in \mathbb{N} \text{ and } m \in \mathbb{Z}. \] For example, we can see that

\[ \Gamma_{1,-1,2} = \begin{pmatrix} 1 & n_2 & \frac{n_2^2}{2} & \frac{n_4}{2} \\ 0 & 1 & n_2 & n_3 \\ 0 & 0 & 1 & n_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad | \quad n_1, n_2, n_3, n_4 \in \mathbb{Z}. \]

We remark also that \( \Gamma_{k,m,n} \cong \Gamma_{k',m',n'} \) if and only if

\[ k = k', \quad n = n', \quad m' = \pm m \mod (k,n), \]

see for example [1, Corollary 6.2.7].

### 4.2. Automorphisms of \( \Gamma_{k,m,n} \)

Let \( \Gamma = \Gamma_{k,m,n} \) and let \( h : \Gamma \to \Gamma \) be an automorphism of \( \Gamma \). Then \( h \) preserves the diagram (4.1). This implies that the diagram (4.1) induces the following commutative diagram

\[
\begin{array}{ccc}
\text{Aut}(\Gamma_k) & \longrightarrow & \text{GL}(2, \mathbb{Z}) \\
\uparrow \vartheta & & \uparrow = \\
\text{Aut}(\Gamma) & \longrightarrow & \text{GL}(2, \mathbb{Z})
\end{array}
\]

Here \( \sigma : \text{Aut}(\Gamma) \to \text{Aut}(\Gamma_k) \) is the homomorphism induced by \( \pi \). We remark also that \( \vartheta \) is surjective and split by Section 2.

Consider the elements \( e_1, e_2, e_3, e_4 \in \text{nil}^4 \)

\[ e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & \frac{m}{k} \pi + \frac{1}{2k} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 0 \frac{1}{k} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

Then

\[ \exp e_1 = \alpha, \quad \exp e_2 = \beta, \quad \exp e_3 = \gamma, \quad \exp e_4 = \delta \]

generate \( \Gamma \), and the nontrivial Lie brackets in \( \text{nil}^4 \) between \( e_1, e_2, e_3 \) and \( e_4 \) are

\[ [e_1, e_2] = -ke_3 + \left( m + \frac{kn}{2} \right) e_4, \quad [e_2, e_3] = ne_4. \]

Consider a Lie algebra automorphism \( \phi : \text{nil}^4 \to \text{nil}^4 \) of \( \text{nil}^4 \). Then \( \phi \) is a linear transformation of the linear space \( \text{nil}^4 \) preserving all the Lie brackets between the linear basis \( \{e_1, e_2, e_3, e_4\} \) of \( \text{nil}^4 \). Because \( \phi \) must preserve the lower central series of \( \text{nil}^4 \), it is of the form

\[ \phi(e_1) = a_{11}e_1 + a_{12}e_2 + p_{11}e_3 + p_{21}e_4, \]
\[ \phi(e_2) = a_{12}e_1 + a_{22}e_2 + p_{12}e_3 + p_{22}e_4, \]
\[ \phi(e_3) = a_{33}e_3 + a_{43}e_4, \]
\[ \phi(e_4) = a_{44}e_4. \]

Because \( \varphi \) must preserve all the Lie brackets (4.5) including the trivial ones, we obtain that
\[ a_{21} = 0, \]
\[ a_{33} = a_{11}a_{22}, \]
\[ a_{44} = a_{22}a_{33} = a_{11}a_{22}^2, \]
\[ a_{43}k = a_{22}p_{11}n + a_{11}a_{22}(a_{22} - 1) \left( m + \frac{kn}{2} \right). \]

Consequently, we have that \( \text{Aut}(\text{Nil}^4) \) is isomorphic to a subgroup of the matrix group \( \text{GL}(4, \mathbb{R}) \):

\[
\begin{pmatrix}
a_{11} & a_{12} & 0 & 0 \\
0 & a_{22} & 0 & 0 \\
p_{11} & p_{12} & a_{11}a_{22} & 0 \\
p_{21} & p_{22} & p_{23} & a_{11}a_{22}^2
\end{pmatrix}
\left\{ \begin{array}{c} \text{p}_{23}k = a_{22}p_{11}n \\
+ a_{11}a_{22}(a_{22} - 1) \left( m + \frac{kn}{2} \right) \end{array} \right\}.
\]

By a result of Mal'cev again, we can regard
\[ \text{Aut}(\Gamma) \subset \text{Aut}(\text{Nil}^4) = \text{Aut}(\text{Nil}^4) \]
via the following commutative diagram

\[
\begin{array}{ccc}
\text{nil}^4 & \xrightarrow{\phi} & \text{nil}^4 \\
\log \downarrow & & \exp \downarrow \\
\text{Nil}^4 & \xrightarrow{\varphi} & \text{Nil}^4 \\
\uparrow & & \uparrow \\
\Gamma & \xrightarrow{\varphi} & \Gamma
\end{array}
\]

**Definition 4.1.** Denote by \( \text{UT}(2, \mathbb{R}) \) the subgroup of \( \text{GL}(2, -) \) consisting of upper triangular matrices.

Thus we have commutative diagrams

\[
\begin{array}{ccc}
\text{Aut}(\text{nil}^4) & \longrightarrow & \text{UT}(2, \mathbb{R}) \\
\uparrow & & \uparrow \\
\text{Aut}(\Gamma) & \xrightarrow{\varphi} & \text{UT}(2, \mathbb{Z})
\end{array}
\]

\[ \longrightarrow 1 \]
and
\[\begin{array}{c}
\text{Aut}(\Gamma_k) \xrightarrow{\vartheta} \text{GL}(2, \mathbb{Z}) \\
\uparrow \sigma \quad \quad \quad \quad \quad \uparrow \sigma
\end{array}\]
\[\begin{array}{c}
\text{Aut}(\Gamma_{k,m,n}) \xrightarrow{\Theta} \text{UT}(2, \mathbb{Z}) \\
\uparrow \quad \quad \quad \quad \quad \uparrow
\end{array}\]
\[\begin{array}{c}
1 \xrightarrow{} \ker(\vartheta) \xrightarrow{} \vartheta^{-1}(\text{UT}(2, \mathbb{Z})) \xrightarrow{\vartheta} \text{UT}(2, \mathbb{Z}) \xrightarrow{} 1
\end{array}\]

This induces the following commutative diagram
\[\begin{array}{c}
1 \xrightarrow{} \ker(\vartheta) \xrightarrow{} \vartheta^{-1}(\text{UT}(2, \mathbb{Z})) \xrightarrow{\vartheta} \text{UT}(2, \mathbb{Z}) \xrightarrow{} 1
\end{array}\]
\[\begin{array}{c}
1 \xrightarrow{} \ker(\Theta) \xrightarrow{} \text{Aut}(\Gamma_{k,m,n}) \xrightarrow{\Theta} \text{UT}(2, \mathbb{Z}) \xrightarrow{} 1
\end{array}\]

Remark that the top extension \(1 \rightarrow \ker(\vartheta) \rightarrow \vartheta^{-1}(\text{UT}(2, \mathbb{Z})) \xrightarrow{\vartheta} \text{UT}(2, \mathbb{Z}) \rightarrow 1\) splits by Section 2. If \(\Theta\) splits then \(\vartheta\) splits as well. We will study whether the bottom extension
\[1 \rightarrow \ker(\Theta) \rightarrow \text{Aut}(\Gamma_{k,m,n}) \xrightarrow{\Theta} \text{UT}(2, \mathbb{Z}) \rightarrow 1\]
splits.

4.3. Splitting of \(\text{Aut}(\Gamma_{k,m,n}) \xrightarrow{\Theta} \text{UT}(2, \mathbb{Z})\)

Remark that the subgroup \(\text{UT}(2, \mathbb{Z})\) of \(\text{GL}(2, \mathbb{Z})\) is generated by
\[\rho = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \kappa = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \eta = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

They satisfy the relations
\[\kappa^2 = \eta^2 = 1, \quad \kappa \eta = \eta \kappa, \quad \kappa \rho \kappa^{-1} = \rho^{-1}, \quad \eta \rho \eta^{-1} = \rho.
\]

To discuss whether \(\Theta\) splits, we first lift \(\rho, \kappa\) and \(\eta\) to elements of \(\text{Aut}(\text{nil}^4)\) using (4.6) as follows:

\[R = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ r_{11} & r_{12} & 1 & 0 \\ r_{21} & r_{22} & r_{23} & 1 \end{pmatrix}\] with \(r_{23}k = r_{11}n,\)

\[K = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ k_{11} & k_{12} & -1 & 0 \\ k_{21} & k_{22} & k_{23} & -1 \end{pmatrix}\] with \(k_{23}k = k_{11}n,\)

\[N = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ n_{11} & n_{12} & 1 & 0 \\ n_{21} & n_{22} & n_{23} & -1 \end{pmatrix}\] with \(n_{23}k = -n_{11}n - 2\left(m + \frac{kn}{2}\right).\)
We first find the conditions on $R, N$ and $K$ to generate a subgroup of $\text{Aut}(\text{nil}^4)$ isomorphic to $\text{UT}(2, \mathbb{Z})$. They must satisfy the identities

\[(4.10) \quad K^2 = N^2 = I_2, \quad KN = NK, \quad KRK^{-1} = R^{-1}, \quad NRN^{-1} = R\]

or equivalently they satisfy

\[
r_{11} = r_{21} = r_{23} = 0, \quad k_{11} = k_{21} = k_{23} = 0,
\]

\[
n_{11} = -2r_{12}, \quad n_{12} = -k_{12},
\]

\[
n_{23} = -\frac{2m + kn + n_{11}}{k}, \quad n_{21} = \frac{n_{11}n_{23}}{2}, \quad n_{22} = \frac{n_{12}n_{23}}{2}.
\]

Next we find the conditions on $R, K$ and $N$ to preserve the lattice $\Gamma_{k,m,n}$. For this purpose, we need to recall that a Lie algebra automorphism of $\text{nil}^4$ is regarded as a Lie group automorphism of $\text{Nil}^4$ via the diagram (4.7). In fact, by considering $\text{Nil}^4$ as the Mal’cev completion of $\Gamma_{k,m,n}$, i.e., by considering the elements of $\text{Nil}^4$ as $\alpha r_1 \beta r_2 \gamma r_3 \delta r_4$ for $r_i \in R$, we can see that

\[
R(\alpha) = \alpha, \quad R(\beta) = \alpha \beta r_{k-n_{11}} \delta^{\frac{6m + 7kn - 3mn_{11} - 12r_{23}}{12}}, \quad R(\gamma) = \gamma, \quad R(\delta) = \delta,
\]

\[
K(\alpha) = \alpha^{-1}, \quad K(\beta) = \beta \gamma^{-n_{12} \delta^{\frac{2m + 3n_{12}}{2}}}, \quad K(\gamma) = \gamma^{-1}, \quad K(\delta) = \delta^{-1},
\]

\[
N(\alpha) = \alpha^{-1} \gamma^{n_{11} \delta^{\frac{n_{11}n_{23}}{2}}}, \quad N(\beta) = \beta^{-1} \gamma^{n_{12} \delta^{\frac{n_{12}n_{23} + n}{2}}}, \quad N(\gamma) = \gamma^{n_{23}}, \quad N(\delta) = \delta^{-1}.
\]

Thus $R, K$ and $N$ preserve the lattice $\Gamma_{k,m,n}$ if and only if

\[
n_{11}, \quad n_{12}, \quad n_{23} \in \mathbb{Z},
\]

\[
n_{11} - k, \quad mn_{12}, \quad n_{11}n_{23}, \quad n_{12}n_{23} \in 2\mathbb{Z},
\]

\[
6m + 7kn - 3mn_{11} \in 12\mathbb{Z},
\]

\[
n + n_{23} = -\frac{2m + n_{11}}{k}.
\]

With $n_{11} = k + 2p \ (p \in \mathbb{Z})$ and $n_{12} = q, n_{23} = -r \in \mathbb{Z}$, we see that the identity $n + n_{23} = -\frac{2m + n_{11}}{k}$ reduces to

\[(4.11) \quad r = 2n + \frac{2(m + pn)}{k},
\]

the condition $6m + 7kn - 3mn_{11} \in 12\mathbb{Z}$ reduces to

\[(4.12) \quad 3m + 2kn - 3pn \in 6\mathbb{Z},
\]

and the remaining conditions reduce to the conditions

\[(4.13) \quad qn, \quad rk, \quad qr \in 2\mathbb{Z}.
\]

By (4.11), $rk$ is even. If $n$ is even then by (4.11) and (4.12), $m + rk - 5pn \in 6\mathbb{Z}$, which implies that $m$ must be even.

Consequently, we have proven the following main result.
**Theorem 4.2.** Given $k, n \in \mathbb{N}$ and $m \in \mathbb{Z}$, the natural surjective homomorphism $\Theta : \text{Aut}(\Gamma_{k,m,n}) \to \text{UT}(2, \mathbb{Z})$ splits if and only if there exists an integer $p$ such that

$$3m + 2kn - 3pn \in 6\mathbb{Z}, \quad \frac{m + pn}{k} \in \mathbb{Z}.$$ 

If $n$ is even, then so is $m$.

**Example 4.3.** For the lattice $\Gamma_{1,-1,2}$ of $\text{Nil}^4$, the corresponding homomorphism $\Theta$ cannot split because $n = 2$ is even and $m = -1$ is odd.

Consider the lattice $\Gamma_{3,1,3}$ of $\text{Nil}^4$. The corresponding homomorphism $\Theta$ cannot split, because $3m + 2kn - 3pn = 21 - 9p \notin 6\mathbb{Z}$.

The homomorphism $\Theta$ corresponding to the lattice $\Gamma_{1,-1,3}$ splits because the above conditions are satisfied if we take $p = -1$.

**Acknowledgement.** The authors would like to thank the referee for reading the manuscript thoroughly and correcting some grammar and typos.

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