Crystallographic groups of Sol

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We classify all the closed 3-dimensional orbifolds with Sol-geometry. These are aspherical orbifolds and so their fundamental groups determine the orbifolds completely. Thus we will classify all the crystallographic groups of Sol, together with all the Bieberbach groups, up to isomorphism.

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1 Introduction

We recall the definition of a geometry in the sense of Thurston [19], [21]. Let $X$ be a complete connected, simply connected Riemannian manifold, and let $\text{Isom}(X)$ be the group of isometries of $X$. A pair $(X, \text{Isom}(X))$ is called a geometry if $\text{Isom}(X)$ acts transitively on $X$ and $\text{Isom}(X)$ contains a discrete subgroup $D$ with the coset space $D\backslash X$ of finite volume. According to Thurston, there are 8 kinds of geometries in dimension 3. They are $\mathbb{R}^3$, $\mathbb{H}^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\text{SL}(2, \mathbb{R})$, Nil and Sol.

A question naturally arisen is the problem of the classification of closed 3-orbifolds with a geometric structure modeled on one of these eight types. The classification problem of closed 3-orbifolds with $\mathbb{R}^3$-geometry is known as the 3-dimensional Euclidean space forms problem. It is known that there are ten classes of 3-dimensional Bieberbach groups, [22, Theorems 3.5.5 and 3.5.9], and there are 230 classes of 3-dimensional crystallographic groups.

Let $G$ be a connected, simply connected Lie group. Then $\text{Aff}(G) = G \rtimes \text{Aut}(G)$ is called the affine group of $G$, where the group operation is given by $(g, \alpha)(h, \beta) = (g \cdot \alpha(h), \alpha \beta)$ and $\text{Aff}(G)$ acts on $G$ by $(g, \alpha)z = g \cdot \alpha(z)$.

Let $G$ be a connected, simply connected nilpotent Lie group and let $C$ be any maximal compact subgroup of $\text{Aff}(G)$. Then a discrete cocompact subgroup $\Gamma$ of $G \rtimes C$ is called an almost crystallographic group. The coset space $\Gamma \backslash G$ is an (infra-)nilmanifold, where $\Gamma$ is a torsion-free discrete cocompact lattice of $G$ ($G \rtimes C$, respectively). The maximal compact subgroup $C$ can be chosen so that $G \rtimes C$ equals $\text{Isom}(G)$. Therefore, torsion-free almost crystallographic groups, called almost Bieberbach groups, are exactly the fundamental groups of compact infra-nilmanifolds. Consequently, a closed 3-dimensional manifold has a Nil-geometry if and only if it is an infra-nilmanifold. All the closed 3-orbifolds with Nil-geometry were classified in [3]. It is shown there that there are only 15 kinds of distinct closed 3-dimensional manifolds with Nil-geometry up to Seifert local invariant. Utilizing the ideas in [3], the aim of this paper is to classify all the closed 3-orbifolds with Sol-geometry up to diffeomorphism. These orbifolds are infra-solvorbifolds. Their fundamental groups are called SC-groups and determine the orbifolds completely [7]. Thus we will classify all their fundamental groups.

P. Scott classified in [19] the closed 3-manifolds with Sol-geometry and W. Dunbar provided in [5], [6] a classification of compact orientable 3-dimensional solvorbifolds which fiber over $S^1$ by classifying certain
finite group actions on torus bundles over $S^1$ with Anosov monodromy. However to our best knowledge, the crystallographic groups modeled on Sol have never been studied explicitly.

The paper is organized as follows. First we recall the necessary terminology and basic facts. In particular, we observe that the affine group of Sol imbeds into the affine group $\text{Aff}(\mathbb{R}^3)$. In the following two sections we review the lattices of Sol mainly from [15] and then we review structure theorems for the so-called solvable space forms problem of type $(\mathbb{R})$. We study the structure of SC-groups modeled on Sol (see the next section for its definition) and find out in particular intrinsic actions induced from SC-groups. In Section 5, we study the extensions $\mathbb{Q}$ of $\mathbb{Z}$ by all the subgroups of the maximal compact subgroup of $\text{Aut}(\text{Sol})$. In order to determine the extensions of $\mathbb{Z}$ by $\mathbb{Q}$ with intrinsic actions as abstract kernels, we study the second cohomology groups $H^2(\mathbb{Q}; \mathbb{Z})$ using special 2-cocycles. This turns out to solve systems of matrix equations. This will be done in Section 7 case by case. Section 8 is for summary of our results. In the final section, we will concentrate on SB-groups or equivalently on the closed 3-manifolds with Sol-geometry. We will compare our exposition with some of known expositions.

## 2 Infra-solvmanifolds modeled on Sol

The Lie group Sol is one of the eight geometries that one considers in the study of 3-manifolds. This geometry has the least symmetry of all the eight geometries as the identity component of the stabilizer of a point is trivial [19].

One can describe Sol as a semi-direct product $\mathbb{R}^2 \rtimes \sigma \mathbb{R}$ where $t \in \mathbb{R}$ acts on $\mathbb{R}^2$ via the map

$$\sigma(t) = \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The Lie group Sol can be imbedded into $\text{Aff}(\mathbb{R}^3)$ as

$$\begin{bmatrix} e^t & 0 & 0 & x \\ 0 & e^{-t} & 0 & y \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $x$, $y$ and $t$ are real numbers.

We say that a closed 3-dimensional manifold $M$ has a Sol-geometry if there is a subgroup $\Pi$ of $\text{Isom}(\text{Sol})$ so that $\Pi$ acts freely and properly discontinuously with compact quotient $M = \Pi \backslash \text{Sol}.$

The group of affine automorphisms of Sol is $\text{Aff}(\text{Sol}) = \text{Sol} \rtimes \text{Aut}(\text{Sol})$. Let $K$ be a maximal compact subgroup of $\text{Aut}(\text{Sol})$ and let $E$ be a torsion-free discrete cocompact subgroup of $\text{Sol} \rtimes K \subset \text{Aff}(\text{Sol})$. Then such a group $E$ is called a (genuine) SB-group and the closed manifold $E \backslash \text{Sol}$ is called a 3-dimensional infra-solvmanifold modeled on Sol. A discrete cocompact subgroup of $\text{Sol} \rtimes K$ is called an SC-group.

It is easy to show (see [9, Proposition 2.3]) that $\text{Aut}(\text{Sol})$ is isomorphic to

$$\text{Aut}_1(\text{Sol}) \bigcup \text{Aut}_1(\text{Sol}) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(2.1)

where

$$\text{Aut}_1(\text{Sol}) = \left\{ \begin{bmatrix} \alpha & 0 & \mu \\ 0 & \beta & \nu \\ 0 & 0 & 1 \end{bmatrix} \bigg| \alpha, \beta, \mu, \nu \in \mathbb{R}, \alpha \beta \neq 0 \right\}.$$

We notice that $\text{Aut}_1(\text{Sol}) \cong \mathbb{R}^2 \rtimes \text{GD}(2)$ where $\text{GD}(2)$ is the group of all invertible $2 \times 2$-diagonal matrices, and it acts on $\mathbb{R}^2$ by matrix multiplication. Furthermore, $\text{Aut}(\text{Sol}) \cong \text{Aut}_1(\text{Sol}) \rtimes \mathbb{Z}_2 \cong (\mathbb{R}^2 \rtimes \text{GD}(2)) \rtimes \mathbb{Z}_2$ where the generator of $\mathbb{Z}_2$ acts on $\mathbb{R}^2$ by left multiplication by the matrix $-\omega$ and on $\text{GD}(2)$ by conjugation by the matrix $\omega$ where

$$\omega = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$
Every maximal compact subgroup of $\text{Aut}(\text{Sol})$ is isomorphic to a (maximal compact) subgroup of $GD(2)$, which is

$$O(1) \times O(1) = (\mathbb{Z}_2)^2 = \left\{ \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$ 

Thus we see that the dihedral group $D(4)$ of order 8 generated by

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a maximal compact subgroup of $\text{Aut}(\text{Sol})$. Note also that $D(4)$ has a subgroup generated by $X^2$ and $XY$, and this group is isomorphic to $(\mathbb{Z}_2)^2$.

Recall from [9, Theorem 3.4] and [10, Theorem 3.3] that there are two non-equivalent left invariant Riemannian metrics on Sol, and for those metrics the full isometry groups are isomorphic to $\text{Sol} \rtimes (\mathbb{Z}_2)^2$ and $\text{Sol} \rtimes D(4)$. Let $K$ be the compact subgroup $(\mathbb{Z}_2)^2$ or $D(4)$ of $\text{Aff}(\text{Sol})$. Thus we may assume that $E \subset \text{Sol} \rtimes K = \text{Isom}(\text{Sol})$, and since $(\mathbb{Z}_2)^2 \subset D(4)$, we shall assume in what follows that $K = D(4)$. Consequently, a closed 3-dimensional manifold has a Sol-geometry if and only if it is an infra-solvmanifold. It is known that the infra-solvmanifolds are determined completely by their fundamental groups $E$.

Recalling that $\text{Sol} = \mathbb{R}^2 \rtimes E$, Sol fits in an exact sequence $1 \rightarrow \mathbb{R}^2 \rightarrow \text{Sol} \rightarrow \mathbb{R} \rightarrow 1$ where $\mathbb{R}^2 = [\text{Sol}, \text{Sol}]$. Hence Sol has the structure of a plane bundle over $\mathbb{R}$. All isometries of Sol preserve the bundle structure. In fact, the following describes how $\text{Aut}(\text{Sol})$ acts on Sol (see [9, Proposition 2.3]):

$$\begin{bmatrix} \alpha & 0 & \mu \\ 0 & \beta & \nu \\ 0 & 0 & 1 \end{bmatrix} : \begin{bmatrix} e^t & 0 & 0 & x \\ 0 & e^{-t} & 0 & y \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} e^t & 0 & 0 & \alpha x + \mu (e^t - 1) \\ 0 & e^{-t} & 0 & \beta y + \nu (1 - e^{-t}) \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} : \begin{bmatrix} e^t & 0 & 0 & x \\ 0 & e^{-t} & 0 & y \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} e^{-t} & 0 & 0 & y \\ 0 & e^t & 0 & x \\ 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$ (2.2) (2.3)

It is easy to see that the above actions by the elements

$$\begin{bmatrix} \alpha & 0 & \mu \\ 0 & \beta & \nu \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$
on Sol are the conjugations by the elements

$$\begin{bmatrix} \alpha & 0 & 0 & -\mu \\ 0 & \beta & 0 & \nu \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

respectively. Therefore, the generators $X, Y$ of $D(4) \subset \text{Aut}(\text{Sol})$ can be viewed as elements of $\text{Aff}(\mathbb{R}^3)$:

$$X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$ (2.4)

Furthermore, the whole affine group of Sol can be imbedded into $\text{Aff}(\mathbb{R}^3)$ as follows:
Proposition 2.1 The affine group of Sol, \( \text{Aff}(\text{Sol}) = \text{Sol} \rtimes \text{Aut}(\text{Sol}) \), imbeds into \( \text{Aff}(\mathbb{R}^3) \subset \text{GL}(4, \mathbb{R}) \), which is given by:

\[
\begin{pmatrix}
e^t & 0 & 0 & x \\
0 & e^{-t} & 0 & y \\
0 & 0 & 1 & t \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\alpha & 0 & \mu \\
0 & \beta & \nu \\
0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
e^t & 0 & 0 & x \\
0 & e^{-t} & 0 & y \\
0 & 0 & 1 & t \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\alpha & 0 & 0 & -\mu \\
0 & \beta & 0 & \nu \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
e^t & 0 & 0 & x \\
0 & e^{-t} & 0 & y \\
0 & 0 & 1 & t \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
e^t & 0 & 0 & x \\
0 & e^{-t} & 0 & y \\
0 & 0 & 1 & t \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
e^t & 0 & 0 & x \\
0 & e^{-t} & 0 & y \\
0 & 0 & 1 & t \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & e^t & 0 & x \\
0 & e^{-t} & 0 & y \\
0 & 0 & 1 & t \\
0 & 0 & 0 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & e^t & 0 & x \\
0 & e^{-t} & 0 & y \\
0 & 0 & 1 & t \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

3 The lattices of Sol

Associated to \( \text{Isom}(\text{Sol}) \), there is an exact commutative diagram

\[
\begin{array}{cccccc}
1 & 1 \\
\downarrow & \downarrow \\
\mathbb{R}^2 & \mathbb{R}^2 \\
\downarrow & \downarrow \\
1 & \longrightarrow & \text{Sol} & \longrightarrow & \text{Isom}(\text{Sol}) & \longrightarrow & K & \longrightarrow & 1 \\
\downarrow & \downarrow & \downarrow & = \\
1 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{R} \times K & \longrightarrow & K & \longrightarrow & 1 \\
\downarrow & \downarrow & \downarrow \\
1 & 1 & 1
\end{array}
\]

The middle, vertical exact sequence is split. Notice also that every automorphism \( \theta \in \text{Aut}(\text{Sol}) \) induces an automorphism \( \tilde{\theta} \) on \( \mathbb{R}^2 = [\text{Sol}, \text{Sol}] \) and in turn an automorphism \( \bar{\theta} \) on \( \mathbb{R} = \text{Sol}/[\text{Sol}, \text{Sol}] \) so that the following diagram is commutative:

\[
\begin{array}{cccccc}
1 & \longrightarrow & \mathbb{R}^2 & \longrightarrow & \text{Sol} & \longrightarrow & \mathbb{R} & \longrightarrow & 1 \\
\downarrow & \bar{\theta} & \downarrow & \bar{\theta} & \downarrow & \bar{\theta} \\
1 & \longrightarrow & \mathbb{R}^2 & \longrightarrow & \text{Sol} & \longrightarrow & \mathbb{R} & \longrightarrow & 1
\end{array}
\]
This is how the maximal compact subgroup $K$ of $\text{Aut(Sol)}$ acts on $\mathbb{R}^2$ and on $\mathbb{R}$. Namely, the $K$-actions are given by the following homomorphisms

$$
\begin{aligned}
K &\xrightarrow{\text{inc.}} \text{Aut(Sol)} &\rightarrow &\xrightarrow{} &\text{Aut}([\mathbb{R}^2]), \\
&\downarrow &\rightarrow &\downarrow &\hat{\theta} \\
&\text{Aut}([\mathbb{R}]) &\rightarrow &\hat{\theta}
\end{aligned}
$$

Therefore, $\mathbb{R} \times K$ acts on $\mathbb{R}^2$ and on $\mathbb{R}$ as follows: For $(r, \theta) \in \mathbb{R} \times K$, $x \in \mathbb{R}^2$ and $s \in \mathbb{R}$,

$$(r, \theta) \cdot x = \sigma(r)\hat{\theta}(x), \quad (r, \theta) \cdot s = \hat{\theta}(s).$$

Let $\Gamma$ be a lattice of $\text{Sol}$. Then $\mathbb{R}^2 \cap \Gamma$ is a lattice of $\mathbb{R}^2$ and $\Gamma/\mathbb{R}^2 \cap \Gamma$ is a lattice of $\text{Sol}/\mathbb{R}^2 = \mathbb{R}$, so that $\mathbb{R}^2 \cap \Gamma \cong \mathbb{Z}^2$ and $\Gamma/\mathbb{R}^2 \cap \Gamma \cong \mathbb{Z}$, and the following diagram of short exact sequences is commutative

$$
\begin{array}{cccc}
1 & \rightarrow & \mathbb{R}^2 & \rightarrow & \text{Sol} & \rightarrow & \mathbb{R} & \rightarrow & 1 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
1 & \rightarrow & \mathbb{Z}^2 & \rightarrow & \Gamma & \rightarrow & \mathbb{Z} & \rightarrow & 1
\end{array}
$$

Choose a basis $\{x_1, x_2\}$ for $\mathbb{Z}^2$ and a basis $t_0$ for $\mathbb{Z}$. Then

$$\sigma(t_0)(x_i) = \ell_{i1}x_1 + \ell_{i2}x_2, \quad (i = 1, 2)$$

for some integers $\ell_{ij}$. Thus the lattice $\Gamma$ is a subgroup of $\text{Sol}$ generated by $x_1, x_2$ and $t_0$ satisfying (3.3). We shall denote such a lattice by

$$\Gamma = \langle x_1, x_2, t_0 | [x_1, x_2] = 1, \sigma(t_0)(x_i) = \ell_{i1}x_1 + \ell_{i2}x_2, (i = 1, 2) \rangle.$$  

Let $P = [x_1, x_2]$ be the matrix with columns $x_1$ and $x_2$, and let

$$A = \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{bmatrix}.$$ 

Then

$$PAP^{-1} = \sigma(t_0) = \begin{bmatrix} e^{\theta_0} & 0 \\ 0 & e^{-\theta_0} \end{bmatrix}$$

and so $A \in \text{SL}(2, \mathbb{Z})$. Note that $P^{-1}$ consists of eigenvectors of $A$ with eigenvalues $e^{\theta_0}$ and $e^{-\theta_0}$. Notice also that $A$ has trace $e^{\theta_0} + e^{-\theta_0} = \ell_{11} + \ell_{22} > 2$. This implies that $A$ is a hyperbolic matrix; it has different real eigenvalues: one is greater than 1 and the other is less than 1. Furthermore, neither $\ell_{12}$ nor $\ell_{21}$ vanishes, see for example [15].

The notation (3.4) for a lattice of $\text{Sol}$ depends on the choice of generators $x_1, x_2$ and $t_0$. For different choices of generators, they are related as follows:

**Lemma 3.1** If a lattice $\Gamma$ of $\text{Sol}$ satisfies

$$\Gamma = \langle x_1, x_2, t_0 | [x_1, x_2] = 1, \sigma(t_0)(x_i) = \ell_{i1}x_1 + \ell_{i2}x_2, (i = 1, 2) \rangle$$

$$= \langle x_1', x_2', t_0' | [x_1', x_2'] = 1, \sigma(t_0')(x_i') = \ell_{i1}'x_1' + \ell_{i2}'x_2', (i = 1, 2) \rangle$$

where $[\ell_{ij}]$ and $[\ell_{ij}']$ are in $\text{SL}(2, \mathbb{Z})$ with trace $> 2$, then for some $R \in \text{GL}(2, \mathbb{Z})$ and $\delta = \pm 1$,

$$[x_1', x_2'] = [x_1, x_2]^R, \quad t_0' = \delta t_0, \quad [\ell_{ij}'] = R^{-1}[\ell_{ij}]^R R.$$

Moreover, the converse also holds.

**Proof.** Since $\{x_1, x_2\}$ and $\{x_1', x_2'\}$ are bases for $\mathbb{Z}^2$, we have

$$[x_1', x_2'] = [x_1, x_2]^R.$$
for some \( R \in \text{GL}(2, \mathbb{Z}) \). Since \( t_0 \) and \( t'_0 \) generators of the lattice \( \mathbb{Z} \) of the quotient group \( \mathbb{R} \cong \text{Sol}/\mathbb{R}^2 \), we must have \( t'_0 = \pm t_0 \). Recall that

\[
[x_1, x_2][\ell_{ij}][x_1, x_2]^{-1} = \sigma(t_0), \quad [x'_1, x'_2][\ell'_{ij}][x'_1, x'_2]^{-1} = \sigma(t'_0).
\]

Since \( \sigma(-t_0) = \sigma(t_0)^{-1} \), we have

\[
[x_1, x_2][\ell_{ij}]^{\pm 1}[x_1, x_2]^{-1} = \left([x_1, x_2][\ell_{ij}][x_1, x_2]^{-1}\right)^{\pm 1} = [x_1, x_2][\ell'_{ij}][x'_1, x'_2]^{-1} = [x_1, x_2] (R[\ell'_{ij}]R^{-1}) [x_1, x_2]^{-1}
\]

and so \( [\ell'_{ij}] = R^{-1}[\ell_{ij}]^{\pm 1} R \). The converse part is trivial.

Let \( \theta : \Gamma \to \Delta \) be an isomorphism between lattices of \( \text{Sol} \). Since \( \text{Sol} \) is a connected and simply connected solvable Lie group of type \((\mathbb{R})\), the isomorphism \( \theta \) extends uniquely to an automorphism \( \hat{\theta} : \text{Sol} \to \text{Sol} \), see for example [12] or [14]. Then it induces automorphisms \( \hat{\theta} : \mathbb{R}^2 \to \mathbb{R}^2 \) and \( \bar{\theta} : \mathbb{R} \to \mathbb{R} \) so that the following diagram is commutative

\[
\begin{array}{cccc}
1 & \longrightarrow & \mathbb{R}^2 & \longrightarrow & \text{Sol} & \longrightarrow & \mathbb{R} & \longrightarrow & 1 \\
\big| \hat{\theta} & & \big| \phi & & \big| \hat{\phi} & & \big| \bar{\theta} & & \big| \bar{\phi} \\
1 & \longrightarrow & \mathbb{R}^2 & \longrightarrow & \text{Sol} & \longrightarrow & \mathbb{R} & \longrightarrow & 1
\end{array}
\]

which yields that for all \( x \in \mathbb{R}^2 \),

\[
\hat{\theta}(\sigma(t_0)(x)) = \sigma(\phi(t_0))(\bar{\phi}(x)).
\]

**Lemma 3.2** Let \( \Gamma \) and \( \Delta \) be isomorphic lattices of \( \text{Sol} \) such that

\[
\Gamma = \left\langle x_1, x_2, t_0 \mid [x_1, x_2] = 1, \sigma(t_0)(x_i) = \ell_{i1}x_1 + \ell_{i2}x_2, \ (i = 1, 2) \right\rangle,
\]

\[
\Delta = \left\langle y_1, y_2, t_1 \mid [y_1, y_2] = 1, \sigma(t_1)(y_i) = m_{i1}y_1 + m_{i2}y_2, \ (i = 1, 2) \right\rangle,
\]

where \( [\ell_{ij}], [m_{ij}] \in \text{SL}(2, \mathbb{Z}) \) with \( \text{trace} > 2 \). Then for some \( R \in \text{GL}(2, \mathbb{Z}) \) and for some nonzero real numbers \( \alpha \) and \( \beta \),

\[
t_1 = \pm t_0, \quad [m_{ij}] = R^{-1}[\ell_{ij}]^{\pm 1} R,
\]

\[
[y_1, y_2] = \begin{cases}
\begin{bmatrix}
\alpha & 0 \\
0 & \beta
\end{bmatrix} [x_1, x_2] R & \text{when the above two } \pm \text{ signs agree,} \\
\begin{bmatrix}
\alpha & 0 \\
0 & \beta
\end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} [x_1, x_2] R & \text{when the } \pm \text{ signs disagree.}
\end{cases}
\]

Moreover, the converse also holds.

**Proof.** Let \( \theta : \Gamma \to \Delta \) be an isomorphism. Since \( \sigma(t_0)(x_i) = \ell_{i1}x_1 + \ell_{i2}x_2 \ (i = 1, 2) \), by taking \( \hat{\theta} \) on both sides, we have

\[
\sigma(\hat{\theta}(t_0))(\hat{\theta}(x_i)) = \hat{\theta}(\sigma(t_0)(x_i)) = \ell_{i1}\hat{\theta}(x_1) + \ell_{i2}\hat{\theta}(x_2) \quad (i = 1, 2).
\]

Thus

\[
\Delta = \theta(\Gamma) = \left\langle \hat{\theta}(x_1), \hat{\theta}(x_2), \hat{\theta}(t_0) \mid [\hat{\theta}(x_1), \hat{\theta}(x_2)] = 1, \sigma(\hat{\theta}(t_0))(\hat{\theta}(x_i)) = \ell_{i1}\hat{\theta}(x_1) + \ell_{i2}\hat{\theta}(x_2), \ (i = 1, 2) \right\rangle.
\]

By Lemma 3.1, for some \( R \in \text{GL}(2, \mathbb{Z}) \),

\[
[y_1, y_2] = \left[\hat{\theta}(x_1), \hat{\theta}(x_2)\right] R, \quad t_1 = \delta\hat{\theta}(t_0), \quad [m_{ij}] = R^{-1}[\ell_{ij}]^{\pm 1} R.
\]
Since \( \bar{\theta} = \pm 1 \), we have
\[
\begin{bmatrix}
y_1 & y_2
\end{bmatrix} = \hat{\theta} \begin{bmatrix} x_1 & x_2 \end{bmatrix} R, \quad t_1 = \text{sign}(\bar{\theta}) \delta t_0, \quad \begin{bmatrix} m_{ij} \end{bmatrix} = R^{-1} \begin{bmatrix} \ell_{ij} \end{bmatrix} \delta R.
\]

By (2.1), \( \hat{\theta} \) is one of the following forms
\[
\begin{bmatrix}
\alpha & 0 \\
0 & \beta
\end{bmatrix} \quad \text{(when sign}(\bar{\theta}) = 1), \quad \begin{bmatrix}
0 & \alpha \\
\beta & 0
\end{bmatrix} \quad \text{(when sign}(\bar{\theta}) = -1).
\]

where \( \alpha \beta \neq 0 \). This finishes the first half of our assertion. For the second half, i.e., for the converse, we can see easily that the automorphism \( \theta \) on Sol (see (2.1))
\[
\theta = \begin{bmatrix}
\alpha & 0 & 0 \\
0 & \beta & 0 \\
0 & 0 & 1
\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}
0 & \alpha & 0 \\
\beta & 0 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]
maps the lattice \( \Gamma \) onto the lattice \( \Delta \). \( \square \)

Motivated from Lemmas 3.1 and 3.2, we introduce an equivalence relation.

**Definition 3.3** An integer matrix \( A \) is said to be weakly conjugate to an integer matrix \( B \), written \( A \sim_w B \), if \( B \) is \( \text{GL}(n, \mathbb{Z}) \)-conjugate to \( A \) or \( A^{-1} \).

Note that \( \sim_w \) is an equivalence relation on \( \text{SL}(2, \mathbb{Z}) \) as well as on the subset of all integer matrices with trace \( > 2 \). Remark also that \( A^{-1} \) is conjugate to \( A \) or \( A' \). In particular, \( A \) is weakly conjugate to \( A^{-1} \) and \( A' \).

Recall for example from [15, Lemma 2.1] that the set of all lattices of Sol is the same as the set of all hyperbolic \( 2 \times 2 \)-integer matrices with trace \( > 2 \). We can refine this relation further. From Lemmas 3.1 and 3.2, we have a well-defined function from the set of all isomorphic lattices of Sol into the set of all weak conjugacy classes of hyperbolic \( 2 \times 2 \)-integer matrices with trace \( > 2 \). In fact, this function is bijective.

**Theorem 3.4** There is a one-to-one correspondence between the set of lattices of Sol modulo isomorphism and the subset of \( \text{SL}(2, \mathbb{Z}) \) with trace \( > 2 \) modulo weak conjugacy.

**Proof.** We shall show that the above function has inverse. Let \( A \) be an integer matrix
\[
A = \begin{bmatrix}
\ell_{11} & \ell_{12} \\
\ell_{21} & \ell_{22}
\end{bmatrix}
\]
with determinant 1 and trace \( \ell_{11} + \ell_{22} > 2 \). Then \( A \) has two distinct irrational eigenvalues \( \frac{1}{2}(\ell_{11} + \ell_{22} \pm \sqrt{(\ell_{11} + \ell_{22})^2 - 4}) \). With two corresponding eigenvectors, we form real invertible matrices
\[
P_0 = \begin{bmatrix}
\frac{\ell_{11} - \ell_{22} + \sqrt{(\ell_{11} + \ell_{22})^2 - 4}}{2\ell_{21}} & \frac{\ell_{11} - \ell_{22} - \sqrt{(\ell_{11} + \ell_{22})^2 - 4}}{2\ell_{21}} \\
\frac{1}{2}
\end{bmatrix}^{-1} = \begin{bmatrix}
x_1 & x_2 \\
y_1 & y_2
\end{bmatrix}.
\]

Let \( t_0 = \ln \frac{\ell_{11} + \ell_{22} + \sqrt{(\ell_{11} + \ell_{22})^2 - 4}}{2} \). Then
\[
P_0 A P_0^{-1} = \begin{bmatrix}
e^{t_0} & 0 \\
0 & e^{-t_0}
\end{bmatrix} = \sigma(t_0).
\]

We consider the lattice of \( \mathbb{R}^2 \) spanned by the linearly independent vectors
\[
x_1 = \begin{bmatrix}
x_1 \\
y_1
\end{bmatrix}, \quad x_2 = \begin{bmatrix}
x_2 \\
y_2
\end{bmatrix}.
\]
Then it is clearly a \( \sigma(t_0) \)-invariant lattice. In fact, we see that
\[
\sigma(t_0)\(x_i\) = \ell_1x_1 + \ell_2x_2.
\]
Hence from \( A \), we realize a lattice \( \Gamma \) of Sol as follows:
\[
\Gamma = \langle x_1, x_2, t_0 \mid [x_1, x_2] = 1, \sigma(t_0)\(x_i\) = \ell_1x_1 + \ell_2x_2, (i = 1, 2) \rangle.
\]
If we choose another basis \( (x'_1, x'_2) \) for the lattice \( (x_1, x_2) \) of \( \mathbb{R}^2 \), then the matrix \( P' \) with columns \( x'_i \) and \( x''_i \) differ from \( P_0 \) by a matrix in \( GL(2, \mathbb{Z}) \), namely, \( P'_0 = P_0R \) for some \( R \in GL(2, \mathbb{Z}) \). Hence
\[
\sigma(t_0) = P_0AP_0^{-1} = (P_0R)(R^{-1}AR)(P_0R)^{-1} = P'_0(R^{-1}AR)P_0^{-1}
\]
and so it gives rise to a realization of the lattice \( \Gamma \) of Sol as
\[
\Gamma = \langle x'_1, x'_2, t_0 \mid [x'_1, x'_2] = 1, \sigma(t_0)\(x'_i\) = \ell'_1x'_1 + \ell'_2x'_2, (i = 1, 2) \rangle
\]
where
\[
A' = R^{-1}AR = \begin{bmatrix} \ell'_1 & \ell'_1 \\ \ell''_1 & \ell''_2 \end{bmatrix}
\]
We notice now that this realization depends only on the choice of eigenvectors of \( A \). We will show that if we choose different eigenvectors of \( A \) then we obtain isomorphic lattice realized into Sol. We choose different eigenvectors of \( A \). The different choice of eigenvectors results in different diagonalizing matrix \( P \) or \( P^{-1} \), namely, for some nonzero numbers \( c \) and \( d \),
\[
P = \begin{bmatrix} \ell_{11} - \ell_{22} + \sqrt{(\ell_{11} + \ell_{22})^2 - 4} \frac{c}{2\ell_{21}} & \ell_{11} - \ell_{22} - \sqrt{(\ell_{11} + \ell_{22})^2 - 4} \frac{d}{2\ell_{21}} \\ \frac{2\ell_{21}}{c} & \frac{2\ell_{21}}{d} \end{bmatrix}^{-1}
\]
\[
= \left(P_0^{-1}\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}\right)^{-1} = \begin{bmatrix} 1/c & 0 \\ 0 & 1/d \end{bmatrix} P_0 := C P_0 = \begin{bmatrix} 1/c & 1/c \\ 1/d & 1/d \end{bmatrix},
\]
or
\[
P = \left(\begin{bmatrix} \ell_{11} - \ell_{22} + \sqrt{(\ell_{11} + \ell_{22})^2 - 4} \frac{1}{2\ell_{21}} & \ell_{11} - \ell_{22} - \sqrt{(\ell_{11} + \ell_{22})^2 - 4} \frac{1}{2\ell_{21}} \\ \frac{2\ell_{21}}{1} & \frac{2\ell_{21}}{1} \end{bmatrix}\right)^{-1}
\]
\[
= C \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} P_0 \quad \text{(coming from the change of order of eigenvectors)}.
\]
In the first case, we obtain a new \( \sigma(t_0) \)-invariant lattice of \( \mathbb{R}^2 \) spanned by
\[
y_1 = \begin{bmatrix} 1/c \ x_1 \\ 1/d \ y_1 \end{bmatrix}, \quad y_2 = \begin{bmatrix} 1/c \ x_2 \\ 1/d \ y_2 \end{bmatrix},
\]
and
\[
PAP^{-1} = CP_0A(CP_0)^{-1} = C\sigma(t_0)C^{-1} = \sigma(t_0).
\]
This yields an isomorphic lattice of Sol given by
\[
\langle y_1, y_2, t_0 \mid [y_1, y_2] = 1, \sigma(t_0)\(y_i\) = \ell_{11}y_1 + \ell_{22}y_2, (i = 1, 2) \rangle.
\]
The second case yields that
\[
PAP^{-1} = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) P_0 A \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) P_0^{-1}
\]
\[
= C \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \left( P_0 A P_0^{-1} \right) \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) C^{-1}
\]
\[
= C \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \sigma(t_0) \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) C^{-1}
\]
\[
= C \sigma(-t_0) C^{-1} = \sigma(-t_0)
\]
and hence we obtain an isomorphic lattice of Sol given by
\[
(y'_1, y'_2, -t_0 | [y_1, y'_2]) = 1, \sigma(-t_0)(y'_i) = \ell_1 y'_1 + \ell_2 y'_2, \quad (i = 1, 2)
\]
where
\[
y'_1 = \left[ \frac{1}{c} y_1 \frac{1}{d} x_1 \right], \quad y'_2 = \left[ \frac{1}{c} y_2 \frac{1}{d} x_2 \right].
\]

In all, we have shown that given a hyperbolic $2 \times 2$-integer matrix with trace $> 2$, there is a canonical way of realizing a lattice of Sol, which is unique up to isomorphism. By the converse parts of Lemmas 3.1 and 3.2, this realization is also unique up to weak conjugacy. Consequently, we have constructed a function from the set of all weak conjugacy classes of hyperbolic $2 \times 2$-integer matrices with trace $> 2$ into the set of all isomorphic lattices of Sol, and from this construction this function is the inverse of the above function.

Remark 3.5 Given a hyperbolic $2 \times 2$-integer matrix $A = [\ell_{ij}]$ with trace $> 2$, we can realize a lattice of Sol of the form given by (3.4). This yields an abstract group, denoted by $\Gamma_A$, with presentation
\[
\Gamma_A = \langle a_1, a_2, t \mid a_1, a_2 = 1, \; t a_1 t^{-1} = a_1 \ell_{11} a_2 \ell_{21}, \; t a_2 t^{-1} = a_1 \ell_{12} a_2 \ell_{22} \rangle
\]
\[
= \langle a_1, a_2, t \mid a_1, a_2 = 1, \; t a_1 t^{-1} = A(a_1), \; t a_2 t^{-1} = A(a_2) \rangle. \tag{3.5}
\]

By Theorem 3.4, we have that $A \sim w B$ if and only if $\Gamma_A \cong \Gamma_B$. In what follows, we shall use the following notations:
\[
t_0 = \ln \frac{\ell_{11} + \ell_{22} + \sqrt{(\ell_{11} + \ell_{22})^2 - 4}}{2},
\]
\[
P_0 = \frac{\ell_{11} - \ell_{22} + \sqrt{(\ell_{11} + \ell_{22})^2 - 4}}{2\ell_{21}} \begin{bmatrix} \frac{1}{\ell_{21}} & \frac{1}{\ell_{21}} \\ \frac{1}{\ell_{21}} & \frac{1}{\ell_{21}} \end{bmatrix}^{-1}
\]
\[
P = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{d} \end{bmatrix}
\]
\[
P_0 = C P_0 = \begin{bmatrix} \frac{1}{c} x_1 & \frac{1}{c} x_2 \\ \frac{1}{d} y_1 & \frac{1}{d} y_2 \end{bmatrix} \quad \text{or} \quad C P_0 = C E P_0 = \begin{bmatrix} \frac{1}{c} y_1 & \frac{1}{c} y_2 \\ \frac{1}{d} x_1 & \frac{1}{d} x_2 \end{bmatrix}
\]
\[(c, d \in \mathbb{R} - \{0\}).
\]

Then $P$ is a diagonalizing matrix of $A$, namely, $PAP^{-1} = \sigma(\pm t_0)$, and any diagonalizing matrix of $A$ must be of the form $P$. The abstract group $\Gamma_A$ can be realized as a lattice of Sol in a various way by taking various $C$’s or $E$. Namely, by taking $C$ or by taking $C$ and $E$, the assignment
\[
a_1 \mapsto \begin{bmatrix} 1 & 0 & 0 & \frac{1}{c} x_1 \\ 0 & 1 & 0 & \frac{1}{d} y_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad a_2 \mapsto \begin{bmatrix} 1 & 0 & 0 & \frac{1}{c} x_2 \\ 0 & 1 & 0 & \frac{1}{d} y_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad t \mapsto \begin{bmatrix} e^{t_0} & 0 & 0 & 0 \\ 0 & e^{-t_0} & 0 & 0 \\ 0 & 0 & 1 & t_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]
or
\[
\begin{bmatrix}
1 & 0 & 0 & \frac{1}{c}y_1 \\
0 & 1 & 0 & \frac{1}{d}x_1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 & 0 & \frac{1}{c}y_2 \\
0 & 1 & 0 & \frac{1}{d}x_2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
e^{-\theta} & 0 & 0 & 0 \\
0 & e^{\theta} & 0 & 0 \\
0 & 0 & 1 & -\theta_0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

realizes \( \Gamma_A \) as a lattice of Sol.

**Lemma 3.6** Every lattice \( \Gamma \) of Sol is centerless.

**Proof.** Every lattice \( \Gamma = \Gamma_A \) of Sol has a presentation (3.5). Thus every element of \( \Gamma_A \) is of the form \( a^m b^n t^k \). Assume \( a^m b^n t^k \in Z(\Gamma_A) \), the center of \( \Gamma_A \). Then we must have \( g(a^m b^n t^k)g^{-1} = a^m b^n t^k \) for all \( g \in \Gamma_A \). Taking \( g = a, b, t \), respectively, we get \( A^k(a) = a, A^k(b) = b \) and \( A^k(a^m b^n) = a^m b^n \). Since \( A \) does not have an eigenvalue 1, the last equality implies that \( m = n = 0 \). The first two equalities imply that \( A^k = I \). However since \( A \) is hyperbolic, it follows that \( k = 0 \). In all, the lattice \( \Gamma_A \) has the trivial center.

**Remark 3.7** Let \( \Gamma = \Gamma_A \) be a lattice of Sol with defining matrix \( A \) as given in (3.5). By \([15, \text{Theorem 2.4}], \) any automorphism \( \theta \) on \( \Gamma_A \) is given by
\[
\theta(a_1) = a_1^{m_1} a_2^{n_1}, \quad \theta(a_2) = a_1^{m_2} a_2^{n_2}, \quad \theta(t) = a_1^p a_2^q t^\zeta,
\]
where
\[
U := \begin{bmatrix}
u_{11} & \nu_{12} \\
u_{21} & \nu_{22}
\end{bmatrix} \in \text{GL}(2, \mathbb{Z}), \quad p, q \in \mathbb{Z} \quad \text{and} \quad \zeta = \pm 1
\]
satisfy the following:

1. if \( \zeta = 1 \), then \( U \) is of the form
\[
U_1 = \begin{bmatrix}u & v \\
u & \ell_{12}
\end{bmatrix} \begin{bmatrix}
u & \ell_{11} - \ell_{22} \\
u & \ell_{21}
\end{bmatrix};
\]
2. if \( \zeta = -1 \), then \( U \) is of the form
\[
U_2 = \begin{bmatrix}-u & \ell_{11} - \ell_{22} \\
u & \ell_{21}
\end{bmatrix} \begin{bmatrix}u & \ell_{12} \\
u & \ell_{21}
\end{bmatrix}.
\]

Observe also that for any diagonalizing matrix \( P \) of \( A \),
\[
PU_1 P^{-1} = \begin{bmatrix}u - \ell_{11} - \ell_{22} + \sqrt{(\ell_{11} + \ell_{22})^2 - 4} \\
0 \\
0 \\
0
\end{bmatrix}
\]
and
\[
PU_2 P^{-1} = \begin{bmatrix}u + \ell_{11} - \ell_{22} + \sqrt{(\ell_{11} + \ell_{22})^2 - 4} \\
0 \\
0 \\
0
\end{bmatrix}
\]

**Lemma 3.8** \( \text{Inn}(\Gamma_A) = \left\{ \begin{bmatrix}A^k & (I - A)p \\
0 & 1
\end{bmatrix} \right\} \quad k \in \mathbb{Z}, \quad p \in \mathbb{Z}^2 \).

**Proof.** It is easy to observe that
\[
\theta \in \text{Inn}(\Gamma_A) \iff \theta(a_1) = A^k(a_1), \quad \theta(t) = (I - A)(a_1^m a_2^n) t
\]
for some integers \( m, n, k \). This proves our Lemma.
4 The structure of SC-groups modeled on Sol

Let $G$ be a connected, simply connected solvable Lie group. Let $C$ be a maximal compact subgroup of the affine group $\text{Aff}(G)$ of $G$. If $\pi$ is a torsion-free discrete cocompact subgroup of $G \rtimes C$, then the coset space $\pi \backslash G$ is called an infra-solvmanifold. If $\pi \subset G$ then the space $\pi \backslash G$ is called a special solvmanifold. We are interested in the case when $G = \text{Sol}$ and we will classify the discrete cocompact subgroups of $\text{Aff}(\text{Sol})$, i.e., SC-groups. The following theorems give rise to global structure theory for discrete cocompact subgroups of $\text{Aff}(G)$.

Theorem 4.1 ([21, Theorem 4.7.13].) A group $\Pi$ is a torsion-free, discrete, cocompact subgroup of $\text{Isom}(\text{Sol})$ if and only if $\Pi$ is torsion-free and contains a lattice $\Gamma \subset \text{Sol}$ of finite index whose centralizer is trivial.

Theorem 4.1 implies that a 3-dimensional closed infra-solvmanifold is finitely covered by a special solvmanifold. In particular, the 3-dimensional closed infra-solvmanifolds are aspherical.

Theorem 4.2 ([14, Theorem A].) Let $G$ be a connected, simply connected solvable Lie group of type (E) and let $C$ be a compact subgroup of $\text{Aut}(G)$. If $G$ has the strong lattice property and if $\Pi$ is a discrete cocompact subgroup of $G \rtimes C$, then $\Gamma = \Pi \cap G$ is a lattice of $G$, and $\Gamma$ has finite index in $\Pi$.

The Lie group $\text{Sol}$ is of type (R) and hence of type (E). It is also easy to observe that $\text{Sol}$ is strongly filtered ([14, Definition 2]) and so has the strong lattice property ([14, Theorem C]). Thus if $\Pi$ is a discrete cocompact subgroup of $\text{Sol} \rtimes K = \text{Sol} \rtimes D(4)$, i.e., if $\Pi$ is an SC-group modeled on $\text{Sol}$, then $\Gamma = \Pi \cap \text{Sol}$ is a lattice of $\text{Sol}$, and $\Gamma$ has finite index in $\Pi$. The finite group $\Phi = \Pi / \Gamma$ is called the holonomy group of $\Pi$.

Theorem 4.3 ([17, Theorem 3.11].) Let $G$ be a connected, simply connected solvable Lie group of type (R), and $C$ be a compact subgroup of $\text{Aut}(G)$. Let $\Pi, \Pi' \subset G \rtimes C$ be discrete cocompact subgroups, which are finite extensions of lattices of $G$. Then every isomorphism $\theta : \Pi \rightarrow \Pi'$ is a conjugation by an element of $G \rtimes \text{Aut}(G)$.

The above theorem has been generalized even further to homomorphisms in [14, Theorem 2.2].

A covering $M \rightarrow M'$ is called essential if no element of the deck transformation group $\Phi$ is homotopic to the identity. The map sending a free homotopy class into $\text{Out}(\pi_1(M))$ defines a natural homomorphism $\rho : \Phi \rightarrow \text{Out}(\pi_1(M))$. The covering is essential if and only if $\rho$ is injective (see [17]).

Theorem 4.4 ([17, Theorem 5.2].) Let $G$ be a connected, simply connected solvable Lie group of type (R), and $\Gamma$ be a lattice of $G$. Then there are only finitely many infra-solvmanifolds which are essentially covered by the special solvmanifold $\Gamma \backslash G$.

Let $M' = \Pi \backslash G$ be an infra-solvmanifold of type (R) which is essentially covered by $M = \Gamma \backslash G$. Then $\Gamma \subset \Pi \subset \text{Aff}(G)$ and the finite deck transformation group $\Phi = \Pi / \Gamma$ injects into $\text{Out}(\Gamma)$. For a fixed abstract kernel $\Phi \subset \text{Out}(\Gamma)$, Theorem 4.4 states that there are only finitely many isomorphism classes of extensions of $\Gamma$ by $\Phi$, realizing the abstract kernel. Furthermore, if $\Pi \cong \Pi'$, then by Theorem 4.3, they are conjugate to each other by an element of $\text{Aff}(\text{Sol})$. This means that the infra-solvmanifolds $\Pi \backslash G$ and $\Pi' \backslash G$ are affinely diffeomorphic. Consequently, up to affine diffeomorphism, there are only finitely many infra-solvmanifolds essentially covered by $M$.

Since our connected, simply connected solvable Lie group $\text{Sol}$ is of type (R), to each lattice $\Gamma$ of $\text{Sol}$ there are only finitely many infra-solvmanifolds which are essentially covered by the solvmanifold $\Gamma \backslash \text{Sol}$.

Let $M = \Pi \backslash \text{Sol}$ be a 3-dimensional infra-solvmanifold. Then the SB-group $\Pi$ is a torsion-free discrete cocompact subgroup of $\text{Isom}(\text{Sol})$. Since $\text{Isom}(\text{Sol}) = \text{Sol} \rtimes K$, we have an exact sequence

$$1 \rightarrow \Gamma \rightarrow \Pi \rightarrow \Pi / \Gamma \rightarrow 1$$

where $\Gamma = \Pi \cap \text{Sol}$ is a lattice of $\text{Sol}$. The finite group $\Phi = \Pi / \Gamma$ (see Theorem 4.1 or 4.2) is the holonomy group of $\Pi$ or $M$. The holonomy group $\Phi$ sits naturally into $K \subset D(4) \subset \text{Aut}(\text{Sol})$.

Regarding the holonomy group $\Pi / \Gamma$, we next recall the following.

Lemma 4.5 ([13, Corollary 3.3].) Let $F$ be a finite group which acts freely on a compact solvmanifold $N$. Then $\text{rk}_p(F) \leq \dim(N)$ for all prime $p$. 

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Since the holonomy group $\Pi/\Gamma$ acts freely on the compact 3-solvmanifold $\Gamma \backslash \text{Sol}$, every $p$-rank of $\Pi/\Gamma$ is at most 3. Since $\Pi/\Gamma \subset D(4)$, we see that

\[
\begin{align*}
\text{rk}_2(\Pi/\Gamma) &= 0, \quad \Pi/\Gamma = 1, \\
\text{rk}_2(\Pi/\Gamma) &= 1, \quad \Pi/\Gamma = \mathbb{Z}_2, \mathbb{Z}_4, \\
\text{rk}_2(\Pi/\Gamma) &= 2, \quad \Pi/\Gamma = (\mathbb{Z}_2)^2, D(4), \\
\text{rk}_2(\Pi/\Gamma) &= 3, \quad \text{none}.
\end{align*}
\]

Therefore, since $D(4) = \langle X, Y \rangle$ with $X^4 = Y^2 = I$ and $XYX^{-1} = Y^{-1}$, we have

\[
\begin{array}{c|c}
\Pi/\Gamma & \text{subgroup(s)} \\
\hline
1 & 1 \\
\mathbb{Z}_2 & \langle Y \rangle, \langle XY \rangle, \langle X^2Y \rangle, \langle X^3Y \rangle, \langle X^4 \rangle, \\
\mathbb{Z}_2 \times \mathbb{Z}_2 & \langle X^2, Y \rangle, \langle X^2, XY \rangle, \\
\mathbb{Z}_4 & \langle X \rangle, \\
D(4) & \langle X, Y \rangle.
\end{array}
\]

The maximal subgroup of $D(4)$ that acts on Sol as orientation-preserving isometries is $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle X^2, XY \rangle$.

Let $\Pi \subset \text{Isom}(\text{Sol})$ be an SC-group with $\Gamma = \Pi \cap \text{Sol}$ and $\Phi = \Pi/\Gamma$. Since, by Theorem 4.2, $\Gamma$ is a lattice of Sol, $\Gamma \cap [\text{Sol}, \text{Sol}]$ is a lattice of $[\text{Sol}, \text{Sol}] = \mathbb{R}^2$, and $\Gamma/\Gamma \cap [\text{Sol}, \text{Sol}]$ is a lattice of $\text{Sol}/[\text{Sol}, \text{Sol}] = \mathbb{R}$. Thus $\Gamma \cap [\text{Sol}, \text{Sol}] \cong \mathbb{Z}^2$ and $\Gamma/\Gamma \cap [\text{Sol}, \text{Sol}] \cong \mathbb{Z}$. Let $Q = \Pi/\mathbb{Z}^2$. Then the diagram (3.1) induces the following commutative diagram:

\[
\begin{array}{cccccc}
1 & \rightarrow & \Gamma & \rightarrow & \Pi & \rightarrow & \Phi & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & Z & \rightarrow & Q & \rightarrow & \Phi & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & 1 & \rightarrow & 1 & \rightarrow & 1 & \rightarrow & 1
\end{array}
\]

The exact sequences $1 \rightarrow Z \rightarrow Q \rightarrow \Phi \rightarrow 1$ in the bottom row and $1 \rightarrow \mathbb{Z}^2 \rightarrow \Pi \rightarrow Q \rightarrow 1$ in the middle column will play a significant role in our discussion (see Section 5 and Sections 7 and 8).

**Notation 4.6** By Proposition 2.1, we have an imbedding $\text{Aff}(\text{Sol})$ into $\text{Aff}(\mathbb{R}^3)$. We denote this imbedding by $\lambda$. Since Sol has subgroups $\mathbb{R}^2$ and $\mathbb{R}$ (see the diagram (3.1))

\[
\mathbb{R}^2 = \begin{Bmatrix}
\begin{bmatrix}1 & 0 & 0 & x \\
0 & 1 & 0 & y \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} | x, y \in \mathbb{R}
\end{Bmatrix} \quad \text{and} \quad \mathbb{R} = \begin{Bmatrix}
\begin{bmatrix}e^t & 0 & 0 & 0 \\
0 & e^{-t} & 0 & 0 \\
0 & 0 & 1 & t \\
0 & 0 & 0 & 1
\end{bmatrix} | t \in \mathbb{R}
\end{Bmatrix},
\]

we have the induced imbedding $\mu = \lambda |_{\mathbb{R} \times \text{Aut}(\text{Sol})} : \mathbb{R} \times \text{Aut}(\text{Sol}) \rightarrow \text{Aff}(\mathbb{R}^3)$ so that the following diagram is commutative.

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Remark 4.7 Given $\theta \in K \subset \text{Aut}(\text{Sol}) \to \text{Aff}(\mathbb{R}^3)$, $\hat{\theta}$ and $\bar{\theta}$ of the diagram (3.2) are induced by conjugating $\theta$:

$\hat{\theta} = \text{conjugation by } \theta : \mathbb{R}^2 \to \mathbb{R}^2,$

$\bar{\theta} = \text{conjugation by } \theta : \mathbb{R} \to \mathbb{R}.$

Since $\Pi \subset \text{Sol} \times K \overset{\lambda}{\to} \text{Aff}(\mathbb{R}^3)$, the elements of $\Pi$ are of the form

$$\begin{bmatrix}
    e^t & 0 & 0 & x \\
    0 & e^{-t} & 0 & y \\
    0 & 0 & 1 & t \\
    0 & 0 & 0 & 1
\end{bmatrix} X^n Y^\ell$$

and since $Q \subset \mathbb{R} \times K \overset{\mu}{\to} \text{Aff}(\mathbb{R}^3)$, the elements of $Q$ are of the form

$$\begin{bmatrix}
    e^t & 0 & 0 & 0 \\
    0 & e^{-t} & 0 & 0 \\
    0 & 0 & 1 & t \\
    0 & 0 & 0 & 1
\end{bmatrix} X^n Y^\ell$$

where $X$ and $Y$ are given by (2.4). From the diagram (4.1), we have homomorphisms $\phi : Q \to \text{Aut}(\mathbb{Z}^2)$ and $\psi : \Phi \to \text{Aut}(\mathbb{Z})$, both induced from conjugation by elements of $Q$, namely, for $q \in Q$ with $\alpha = (r, \theta) \in \Pi \mapsto q \in Q$, we have $\lambda(\alpha) = w \mu(q)$ for some $w \in \mathbb{Z}^2$ and thus

$$\psi(\pi(q)) = \text{conjugation by } \theta = \bar{\theta},$$  \hspace{1cm} (4.2)

$$\phi(q) = \text{conjugation by } \mu(q).$$  \hspace{1cm} (4.3)

These are our key observations. Since we know all the subgroups $\Phi$ of the maximal compact subgroup $K$ of $\text{Aut}(\text{Sol})$, we can determine all the possible homomorphisms $\psi : \Phi \to \text{Aut}(\mathbb{Z})$, and then all the possible extensions $Q$ of $\mathbb{Z}$ by $\Phi$ with abstract kernel $\psi$ (see STEP 1). Next, after finding an imbedding $\mu : Q \to \text{Aff}(\mathbb{R}^3)$, we can determine all the possible homomorphisms $\phi : Q \to \text{Aut}(\mathbb{Z}^2)$, and then all the possible extensions $E$ of $\mathbb{Z}^2$ by $Q$ with abstract kernel $\phi$ (see STEP 2).

Given an integer matrix $A$ with determinant 1 and trace $> 2$ and $\Phi \subset K$, we are required to find all the possible inclusions $\rho : \Phi \hookrightarrow \text{Out}(\Gamma_A)$ (see Theorem 4.4), and then to each inclusion $\rho$, we find the unique extension $\Pi$ realizing $\rho$ as abstract kernel, and the centralizer of $\Gamma_A$ in $\Pi$ is trivial (see Theorem 4.1). The uniqueness of such extension follows from Lemma 3.6. However, it seems impossible to describe $\text{Out}(\Gamma_A)$ explicitly and so the finite subgroups of $\text{Out}(\Gamma_A)$ in general. Therefore, we achieve our classification problem by doing the following procedures.

**STEP 1.** We classify the extensions $1 \to \mathbb{Z} \to Q \to \Phi \to 1$ with abstract kernel $\psi : \Phi \to \text{Aut}(\mathbb{Z})$ of (4.2) (see Section 5).

**STEP 2.** We classify the (torsion-free) extensions $1 \to \mathbb{Z}^2 \to \Pi \to Q \to 1$ with abstract kernel $\phi : Q \to \text{Aut}(\mathbb{Z}^2)$ of (4.3) (see Section 7).

**STEP 3.** We check $\Pi$ has $\Gamma_A$ as a finite index subgroup with the trivial centralizer and $\Pi$ has abstract kernel $\rho : \Phi \to \text{Out}(\Gamma_A)$ which is an inclusion (see Section 8).
5 Extensions $Q$ of $\mathbb{Z}$ by finite subgroups of $D(4)$

Write $D(4) = \{x, y \mid x^4 = y^2 = 1, yxy^{-1} = x^{-1}\}$. The homomorphism $\psi : D(4) \to \text{Aut}(\mathbb{Z})$ of (4.2) is given by $\psi(x) = -1$ and $\psi(y) = 1$. For all nontrivial subgroups $\Phi$ of $D(4)$, we will classify the extensions $1 \to \mathbb{Z} \to Q \to \Phi \to 1$ having $\psi = \psi_1$ as abstract kernel. For this purpose, we compute $H^2_\psi(\Phi, \mathbb{Z})$, and then we simply write out all the possible (inequivalent) presentations for $Q$ corresponding to the elements of $H^2_\psi(\Phi, \mathbb{Z})$.

Applying a construction from [2] we will manufacture a very simple projective resolution for certain semidirect products $G = K \rtimes_{\sigma} H$.

**Definition 5.1** We say that a free resolution $\epsilon : F \to \mathbb{Z}$ of $\mathbb{Z}$ over $\mathbb{Z}[K]$ admits an action of $H$ compatible with $\sigma$ if for all $h \in H$ there is an augmentation-preserving chain map $\tau(h) : F \to F$ satisfying

1. $\tau(h)(k \cdot f) = \sigma(h)(k) \cdot \tau(h)(f)$ for all $k \in K$ and $f \in F$,
2. $\tau(h)\tau(h') = \tau(hh')$ for all $h, h' \in H$,
3. $\tau(1) = 1_F$.

If such an action exists we can give $F$ a $G$-module structure as follows. If $g \in G$, then $g$ can be expressed uniquely as $g = kh$, with $k \in K$ and $h \in H$. We set $g \cdot f = (kh) \cdot f = k \cdot \tau(h)(f)$. Note that given any $H$-module $L$, this inflates to a $G$-action on $L$ via the projection $G \to H$.

**Lemma 5.2**

<table>
<thead>
<tr>
<th>$\Phi$</th>
<th>generator(s)</th>
<th>$H^2_\psi(\Phi, \mathbb{Z})$</th>
<th>$\psi : \Phi \to \text{Aut}(\mathbb{Z})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}_2$</td>
<td>$\langle y \rangle$</td>
<td>$\mathbb{Z}_2$</td>
<td>trivial,</td>
</tr>
<tr>
<td></td>
<td>$\langle x^2y \rangle$</td>
<td>$\mathbb{Z}_2$</td>
<td>trivial,</td>
</tr>
<tr>
<td></td>
<td>$\langle x^3 \rangle$</td>
<td>$\mathbb{Z}_2$</td>
<td>trivial,</td>
</tr>
<tr>
<td></td>
<td>$\langle xy \rangle$</td>
<td>$0$</td>
<td>nontrivial,</td>
</tr>
<tr>
<td></td>
<td>$\langle x^3y \rangle$</td>
<td>$0$</td>
<td>nontrivial,</td>
</tr>
<tr>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_2$</td>
<td>$\langle x^2, y \rangle$</td>
<td>$\mathbb{Z}_2 \oplus \mathbb{Z}_2$</td>
<td>trivial,</td>
</tr>
<tr>
<td></td>
<td>$\langle x^2, xy \rangle$</td>
<td>$\mathbb{Z}_2$</td>
<td>nontrivial,</td>
</tr>
<tr>
<td>$\mathbb{Z}_4$</td>
<td>$\langle x \rangle$</td>
<td>$0$</td>
<td>nontrivial,</td>
</tr>
<tr>
<td>$D(4)$</td>
<td>$\langle x, y \rangle$</td>
<td>$\mathbb{Z}_2$</td>
<td>nontrivial.</td>
</tr>
</tbody>
</table>

**Proof.** We will only deal with the case where $\Phi = D(4)$. For the computation, using the fact that $D(4) = \mathbb{Z}_4 \rtimes_{\sigma} \mathbb{Z}_2 = \langle x \rangle \rtimes_{\sigma} \langle y \rangle$, we construct the following free resolution of $\mathbb{Z}$ over $\mathbb{Z}[D(4)]$, see [2]. Let $F_i$ be a copy of $\mathbb{Z}[\mathbb{Z}_4]$ and $N = 1 + x + x^2 + x^3$. Consider the usual free resolution $(F, d') : \cdots \to F_3 \xrightarrow{x^{-1}} F_2 \xrightarrow{x^{-1}} F_1 \xrightarrow{x^{-1}} F_0 \xrightarrow{e} \mathbb{Z}$, of $\mathbb{Z}$ over $\mathbb{Z}[\mathbb{Z}_4]$. Define $\tau = \tau(y) : F_4 \to F_4$ by multiplication by given element and $\tau(x \cdot a) = x^{-1} \cdot \tau(a)$ for a generator $a$ of $F$. Then we have the following commutative diagram satisfying $\tau^2 = \text{id}_F$.

\[
\begin{array}{ccccccc}
\cdots & \xrightarrow{x^{-1}} & F_4 & \xrightarrow{N} & F_3 & \xrightarrow{x^{-1}} & F_2 & \xrightarrow{N} & F_1 & \xrightarrow{x^{-1}} & F_0 & \to & \mathbb{Z} \\
\downarrow & & 1 & \downarrow & x^{-1} & \downarrow & 1 & \downarrow & -x^{-1} & \downarrow & 1 & \\
\cdots & \xrightarrow{x^{-1}} & F_4 & \xrightarrow{N} & F_3 & \xrightarrow{x^{-1}} & F_2 & \xrightarrow{N} & F_1 & \xrightarrow{x^{-1}} & F_0 & \to & \mathbb{Z}
\end{array}
\]

Thus we have an action of $\mathbb{Z}_2$ on $F_4$, which is compatible with $\sigma : \mathbb{Z}_2 \to \text{Aut}(\mathbb{Z}_4)$. Consider the usual free resolution $(P, d'') : \cdots \xrightarrow{1+y} P_3 \xrightarrow{y^{-1}} P_2 \xrightarrow{y+1} P_1 \xrightarrow{y^{-1}} P_0 \xrightarrow{e} \mathbb{Z}$, of $\mathbb{Z}$ over $\mathbb{Z}[\mathbb{Z}_2]$. Then we can write out the resolution of $\mathbb{Z}$ over $\mathbb{Z}[D(4)]$ explicitly as follows:

\[
(F \otimes P) = \bigoplus_{p+q=n} F_p \otimes P_q
\]
and $F \otimes P$ is a copy of $\mathbb{Z}[D(4)]$ with generator $a_p \otimes b_q$. The differential is given, as usual, by
\[ d(a_p \otimes b_q) = (d'a_p) \otimes b_q + (-1)^p a_p \otimes (d''b_q). \]
Thus $(F \otimes P, d)$ is a free resolution for $D(4)$ by [2, Proposition 1.2].

Now we can show that
\[ \text{Hom}_{D(4)}((F \otimes P)_1, \mathbb{Z}) \rightarrow \text{Hom}_{D(4)}((F \otimes P)_2, \mathbb{Z}), \]
\[ (m, n) \in \mathbb{Z}^2 \rightarrow (2m, -2m, 0) \in \mathbb{Z}^3, \]
\[ \text{Hom}_{D(4)}((F \otimes P)_2, \mathbb{Z}) \rightarrow \text{Hom}_{D(4)}((F \otimes P)_3, \mathbb{Z}), \]
\[ (m, n, k) \in \mathbb{Z}^3 \rightarrow (0, -2m - 2n, 0, -2k) \in \mathbb{Z}^4. \]
Thus, $H_q^2(D(4), \mathbb{Z}) = \{(m, -m, 0)/(2m, -2m, 0)\} \cong \mathbb{Z}_2$. \(\square\)

Let $\Phi$ be a subgroup of $D(4)$ with generators $\alpha_1$ (and $\alpha_2$). For the simplicity of notation only, we assume $\Phi$ has two generators $\alpha_1, \alpha_2$. For each element $\alpha$ of $\Phi$ we fix a unique word $u(\alpha) = \alpha_1 \cdot \alpha_2 \cdots \alpha_n$ which represents it. So,
\[ \Phi = \{\alpha_1, \alpha_2 | w_i(\alpha_1, \alpha_2) = 1 \ (1 \leq i \leq p)\}. \]
Then, an extension $Q$ of $\mathbb{Z}$ by $\Phi$ compatible with $\psi$ can be presented as
\[ Q = \{t, \alpha_1, \alpha_2 | w_i(\alpha_1, \alpha_2) = r^i \ (1 \leq i \leq p), \alpha_i t \alpha_i^{-1} = \psi(t)(i = 1, 2)\} \] (5.1)
and the elements $q$ of $Q$ can be written uniquely as words
\[ q = r^i u(\alpha) \ (q \in \mathbb{Z}, \alpha \in \Phi). \]
This is completely determined by the set of integers $\ell_i$. Without confusion, we will abuse the symbol $\alpha_i$ as elements of both $Q$ and $\Phi$ when $\alpha_i \in Q$ is mapped to $\alpha_i \in \Phi$ under the natural quotient map $Q \rightarrow \Phi$.

With the help of $H_q^2(\psi, \mathbb{Z})$, we can easily find out the possible (inequivalent) extensions $Q$ of $\mathbb{Z}$ by $\Phi$ having $\psi$ as abstract kernel.

**Lemma 5.3** Given the following $\Phi \subset D(4)$ and $\psi: \Phi \subset D(4) \rightarrow \text{Aut}(\mathbb{Z})$, we have:

(1) $\Phi \cong \mathbb{Z}_2 = \{\beta\}, \psi(\beta) = 1$,
   - As $\Phi \subset D(4)$, $\Phi = \langle y, \langle x^2y \rangle, \langle x^3 \rangle \rangle$,
   - $Q_1 = \{t, \beta | \beta^2 = 1, \beta t \beta^{-1} = t\} = \mathbb{Z} \times \mathbb{Z}_2$,
   - $Q_2 = \{t, \beta | \beta^2 = t, t \beta^{-1} = t\} = \mathbb{Z}$,
(2) $\Phi \cong \mathbb{Z}_2 = \{\beta\}, \psi(\beta) = -1$,
   - As $\Phi \subset D(4)$, $\Phi = \langle xy, \langle x^3y \rangle \rangle$,
   - $Q_3 = \{t, \beta | \beta^2 = 1, \beta t \beta^{-1} = t^{-1}\} = \mathbb{Z} \times \mathbb{Z}_2$;
(3) $\Phi \cong \mathbb{Z}_2 \times \mathbb{Z}_2 = \{\alpha, \beta\}, \psi(\alpha) = \psi(\beta) = 1$,
   - As $\Phi \subset D(4)$, $\Phi = \langle x, y \rangle$,
   - $Q_4 = \{t, \alpha, \beta | \alpha^2 = 1, \beta^2 = 1, t \alpha^2 = t, t \beta = t, [\alpha, \beta] = 1\}$
   \[ = \mathbb{Z} \times (\mathbb{Z}_2 \times \mathbb{Z}_2), \]
   - $Q_5 = \{t, \alpha, \beta | \alpha^2 = t, \beta^2 = t, t \alpha^2 = t, t \beta = t, [\alpha, \beta] = 1\}$
   \[ = \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2, \]
   - $Q_6 = \{t, \alpha, \beta | \alpha^2 = t, \beta^2 = t, t \alpha^2 = t, t \beta = t, [\alpha, \beta] = 1\}$
   \[ = \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2, \]
   - $Q_7 = \{t, \alpha, \beta | \alpha^2 = t, \beta^2 = t, t \alpha^2 = t, t \beta^{-1} = t, [\alpha, \beta] = 1\}$
   \[ = \{t, \alpha, \beta | \alpha^2 = t, \beta^2 = t, t \alpha^2 = t, t \beta^{-1} = t, [\alpha, \beta] = 1\}, \]
   - $\Phi \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ where $\beta = \alpha^{-1} \beta$;
(4) $\Phi \cong \mathbb{Z}_2 \times \mathbb{Z}_2 = \{\alpha, \beta\}, \psi(\alpha) = 1, \psi(\beta) = -1$,
   - As $\Phi \subset D(4)$, $\Phi = \langle x, xy \rangle$. 

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\[ Q_8 = \langle t, \alpha, \beta \mid \alpha^2 = 1, \beta^2 = 1, \alpha t \alpha^{-1} = t, \beta t \beta^{-1} = t^{-1}, [\alpha, \beta] = 1 \rangle = (\mathbb{Z} \times \mathbb{Z}_2) \times \mathbb{Z}_2, \]
\[ Q_9 = \langle t, \alpha, \beta \mid \alpha^2 = t, \beta^2 = 1, \alpha t \alpha^{-1} = t, \beta t \beta^{-1} = t^{-1}, [\alpha, \beta] = t \rangle = \mathbb{Z} \times \mathbb{Z}_2; \]

(5) \( \Phi \cong \mathbb{Z}_4 = \langle \alpha \rangle, \psi(\alpha) = -1, \)
\( \Phi \subset D(4), \Phi = \langle x \rangle, \)
\( Q_{10} = \langle t, \alpha \mid \alpha^4 = 1, \alpha t \alpha^{-1} = t^{-1} \rangle = \mathbb{Z} \times \mathbb{Z}_4; \)

(6) \( \Phi = D(4), \Phi = \langle x, y \rangle, \)
\( Q_{11} = \langle t, \alpha, \beta \mid \alpha^4 = 1, \beta^2 = t, \alpha t \alpha^{-1} = t^{-1}, \beta t \beta^{-1} = t, \beta \alpha \beta^{-1} = t \alpha \rangle \)
\( = \langle (t) \times (\alpha) \rangle \times (\alpha \beta^{-1}) = (\mathbb{Z} \times \mathbb{Z}_4) \times \mathbb{Z}_2, \)
\( Q_{12} = \langle t, \alpha, \beta \mid \alpha^4 = 1, \beta^2 = 1, \alpha t \alpha^{-1} = t^{-1}, \beta t \beta^{-1} = t, \beta \alpha \beta^{-1} = \alpha \rangle \)
\( = \mathbb{Z} \times D(4). \)

**Proof.** From Lemma 5.2, the groups \( Q_1, Q_2, Q_3 \) and \( Q_{10} \) are obvious. The remaining \( Q \)'s are almost obvious. For example, we consider \( Q_8 \) and \( Q_9 \). These groups contain \( Q_1 \) or \( Q_2 \) as a subgroup. Thus together with Lemma 5.2, we must have

- \( Q_8 = \langle t, \alpha, \beta \mid \alpha^2 = 1, \beta^2 = 1, \alpha t \alpha^{-1} = t, \beta t \beta^{-1} = t^{-1}, [\alpha, \beta] = t^m \rangle, \)
- \( Q_9 = \langle t, \alpha, \beta \mid \alpha^2 = t, \beta^2 = 1, \alpha t \alpha^{-1} = t, \beta t \beta^{-1} = t^{-1}, [\alpha, \beta] = t^n \rangle. \)

In the case of \( Q_8 \), we have that
\[
\alpha \beta \alpha^{-1} = t^m \beta \Rightarrow \alpha (\alpha \beta \alpha^{-1}) \alpha^{-1} = \alpha (t^m \beta) \alpha^{-1} \Rightarrow \beta = t^m \alpha \beta \alpha^{-1} \Rightarrow 1 = t^m [\alpha, \beta] \Rightarrow m = 0.
\]

In the case of \( Q_9 \), we have that
\[
\alpha \beta \alpha^{-1} = t^n \beta \Rightarrow \alpha (\alpha \beta \alpha^{-1}) \alpha^{-1} = \alpha (t^n \beta) \alpha^{-1} \Rightarrow \beta \alpha t \beta^{-1} = t^n \alpha \beta \alpha^{-1} \Rightarrow t^n [\alpha, \beta] \]
\( \Rightarrow t^2 = t^{2n} \Rightarrow n = 1. \)

**Remark 5.4** Let \( A \) be a hyperbolic 2 \( \times \) 2-integer matrix with trace \( > 2 \). The abstract group
\[ \Gamma_A = \langle a_1, a_2, t \mid [a_1, a_2] = 1, ta_1 t^{-1} = A(a_1), i = 1, 2 \rangle \]
can be realized as a lattice of Sol in a various way. We can imbed it into Sol by choosing diagonalizing matrices \( P \) of \( A \) (see Notation 3.5) or by choosing diagonalizing matrices of weakly conjugate matrices of \( A \) (see Theorem 3.4). Denote by \( \lambda : \Gamma_A \rightarrow \text{Sol} \rightarrow \text{Aff}(\mathbb{R}^3) \) one of such an imbedding. We may assume
\[
\lambda(t) = \begin{bmatrix}
e^0 & 0 & 0 & 0 \\
0 & e^{-b} & 0 & 0 \\
0 & 0 & 1 & t_0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

Under the affine imbedding \( \mu : \mathbb{R} \times K \subset \text{Sol} \times \text{Isom(Sol)} \rightarrow \text{Aff}(\mathbb{R}^3) \), every element \( q \) of the groups \( Q \subset \mathbb{R} \times K \) of Lemma 5.3 can be written as
\[
\mu(q) = \begin{bmatrix}
e^t & 0 & 0 & 0 \\
0 & e^{-t} & 0 & 0 \\
0 & 0 & 1 & t \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}^{n}^{\ell}.
\]

for some real number \( t \), and integers \( n, \ell \).
For each \( i = 1, 2, \ldots, 12 \), we take \( \mu \) to the relations of \( Q_i \) and we obtain the following faithful representation \( \mu_i : Q_i \subset \mathbb{R} \times K \rightarrow \text{Aff}(\mathbb{R}^3) \):

\[
\mu_i(t) = \lambda(t),
\]
\[
\mu_i(\alpha) = \begin{bmatrix} e^{\theta/2} & 0 & 0 & 0 \\ 0 & e^{-\theta/2} & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}^k_1 \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^m_1,
\]
\[
\mu_i(\beta) = \begin{bmatrix} e^{\theta/2} & 0 & 0 & 0 \\ 0 & e^{-\theta/2} & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}^k_2 \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^n_2 \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^\ell_2,
\]

where \( m, n, \ell \) are determined by the images of \( \alpha \) and \( \beta \) under the projection \( \pi : Q_i \rightarrow \Phi \), namely, our \( m, n \) and \( \ell \) satisfy that \( \alpha \rightarrow x^m \) and \( \beta \rightarrow x^n y^\ell \), and the integers \( k_1 \) and \( k_2 \) are determined by the relations; if \( \alpha^2 = t \) then \( k_1 = 1 \), otherwise \( k_1 = 0 \) and if \( \beta^2 = t \) then \( k_2 = 1 \), otherwise \( k_2 = 0 \). For an example let us consider the group \( Q_{11} \). Since \( \alpha \rightarrow x \) and \( \beta \rightarrow y \), and \( \alpha^4 = 1 \) and \( \beta^2 = t \), we define \( \mu_{11} : Q_{11} \rightarrow \text{Aff}(\mathbb{R}^3) \) by taking \( m = 1, k_1 = 0, n = 0, \ell = 1, k_2 = 1 \) and then we can see all the relations are preserved by \( \mu_{11} \).

The following remarks together with Remark 5.4 are very crucial in determining the possible actions \( \phi : Q \rightarrow \text{Aut}(\mathbb{Z}^2) \).

**Remark 5.5** Let \( \gamma \in Q \) with \( \phi(\gamma) = M \in \text{Aut}(\mathbb{Z}^2) \). By Remark 3.7, \( M \) is one of the forms \( U_j \).

**Case 1.** \( \gamma^2 = 1 \) and \( \gamma \gamma^{-1} = t \): Then \( M^2 = I \), \( \xi = 1 \) and so \( M \) is an integer matrix of the form \( U_1 \). This implies that \( M = \pm I \) and \( MAM^{-1} = A \). This gives no restriction on (the weak conjugacy class of) \( A \), and this case occurs when \( Q = Q_1, Q_5, Q_6, Q_7, Q_8, Q_{12} \).

**Case 2.** \( \gamma^2 = t \) and \( \gamma \gamma^{-1} = t \): Then as \( M^2 = A, M \) is a square root of \( A \), and as \( \xi = 1, M \) is of the form \( U_1 \). Also, \( MAM^{-1} = A \). By Remark 3.7, taking conjugation by \( P \), we have

\[
PMP^{-1} = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}
\]

and

\[
\begin{bmatrix} e^{\theta} & 0 \\ 0 & e^{-\theta} \end{bmatrix} = PAP^{-1} = PM^2P^{-1} = \begin{bmatrix} d_1^2 & 0 \\ 0 & d_2^2 \end{bmatrix}.
\]

This implies that \( d_1 = \pm e^{\theta/2}, d_2 = \pm e^{-\theta/2} \). Recall that

\[
e^{\pm \theta} = \ell_{11} + \ell_{22} \pm \sqrt{(\ell_{11} + \ell_{22})^2 - 4}.
\]

So,

\[
e^{\pm \theta/2} = \frac{\sqrt{\ell_{11} + \ell_{22} + 2 \pm \sqrt{\ell_{11} + \ell_{22} - 2}}}{2}.
\]

Denoting

\[
\sqrt{PAP^{-1}} = \begin{bmatrix} e^{\theta/2} & 0 \\ 0 & e^{-\theta/2} \end{bmatrix}
\]

\[
= \begin{bmatrix} \sqrt{\ell_{11} + \ell_{22} + 2} + \ell_{11} + \ell_{22} - 2 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ \sqrt{\ell_{11} + \ell_{22} + 2} - \ell_{11} + \ell_{22} - 2 \end{bmatrix},
\]
we have that $M$ is equal to
\[ \pm P^{-1} \sqrt{PA} P^{-1} = \pm \begin{bmatrix} \frac{\ell_{11} + 1}{\sqrt{\ell_{11} + \ell_{22} + 2}} & \frac{\ell_{12}}{\sqrt{\ell_{11} + \ell_{22} + 2}} \\ \frac{\ell_{21}}{\sqrt{\ell_{11} + \ell_{22} + 2}} & \frac{\ell_{22} + 1}{\sqrt{\ell_{11} + \ell_{22} + 2}} \end{bmatrix} \]
or
\[ \pm P^{-1} \sqrt{PA} P^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \pm \begin{bmatrix} \frac{\ell_{11} - 1}{\sqrt{\ell_{11} + \ell_{22} - 2}} & \frac{\ell_{12}}{\sqrt{\ell_{11} + \ell_{22} - 2}} \\ \frac{\ell_{21}}{\sqrt{\ell_{11} + \ell_{22} - 2}} & \frac{\ell_{22} - 1}{\sqrt{\ell_{11} + \ell_{22} - 2}} \end{bmatrix}. \]

Note that the first case has determinant 1 and the second case has determinant $-1$. Furthermore, $M$ cannot be both cases because $\sqrt{(\ell_{11} + \ell_{22})^2 - 4}$ is an irrational number. Observe the uniqueness, namely, if $M$ is a square root of $A$ then it is one of the following:
\[ \pm \begin{bmatrix} \frac{\ell_{11} + 1}{\sqrt{\ell_{11} + \ell_{22} + 2}} & \frac{\ell_{12}}{\sqrt{\ell_{11} + \ell_{22} + 2}} \\ \frac{\ell_{21}}{\sqrt{\ell_{11} + \ell_{22} + 2}} & \frac{\ell_{22} + 1}{\sqrt{\ell_{11} + \ell_{22} + 2}} \end{bmatrix} \] (when $\det(M) = 1$),
\[ \pm \begin{bmatrix} \frac{\ell_{11} - 1}{\sqrt{\ell_{11} + \ell_{22} - 2}} & \frac{\ell_{12}}{\sqrt{\ell_{11} + \ell_{22} - 2}} \\ \frac{\ell_{21}}{\sqrt{\ell_{11} + \ell_{22} - 2}} & \frac{\ell_{22} - 1}{\sqrt{\ell_{11} + \ell_{22} - 2}} \end{bmatrix} \] (when $\det(M) = -1$).

Notice also that if $M$ is a square root of $A$, then $QM \pm Q^{-1}$ is a square root of $QA \pm Q^{-1}$. Hence $A$ has a square root in $\text{GL}(2, \mathbb{Z})$ if and only if any weakly conjugate matrix to $A$ has a square matrix in $\text{GL}(2, \mathbb{Z})$. Finally the condition $\text{MAM}^{-1} = A$ is satisfied. Consequently, this case occurs when $A$ has a square root in $\text{GL}(2, \mathbb{Z})$ and when $Q = Q_2, Q_5, Q_6, Q_7, Q_9, Q_{11}$.

We have to notice here that every $A$ does not have a square root in $GL(2, \mathbb{Z})$, and even though $A$ has a square root, its square roots do not have to have trace $> 2$. For example, consider
\[ A_1 = \begin{bmatrix} 12 & 5 \\ 7 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}. \]

Note that these are hyperbolic with trace $> 2$, but it is easy to see that there is no $M \in \text{GL}(2, \mathbb{Z})$ for which $M^2 = A_1$, and the square roots of $A_2$ and $A_3$ in $\text{GL}(2, \mathbb{Z})$ are respectively
\[ \pm \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \] and \[ \pm \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}. \]

Note also that if $\det(M) = 1$ then $|\text{tr}(M)| = \sqrt{\ell_{11} + \ell_{22} + 2} > 2$ as $\ell_{11} + \ell_{22} > 2$.

Case 3. $\gamma^2 = 1$ and $\gamma t \gamma^{-1} = t^{-1}$: Then $M^2 = I$, $\text{MAM}^{-1} = A^{-1}$ and, as $\zeta = -1$, $M$ is of the form $U_2$ and hence $\det(M) = -1$. This case occurs when $Q = Q_3, Q_8, Q_9$.

Case 4. $\gamma^4 = 1$ and $\gamma t \gamma^{-1} = t^{-1}$: Then $M^4 = I$, $\text{MAM}^{-1} = A^{-1}$ and, as $\zeta = -1$, $M$ is of the form $U_2$. This case occurs when $Q = Q_{10}, Q_{11}$.

Recalling from (4.3) that how $Q$ acts on $\mathbb{Z}^2$ and $\mathbb{Z}$, we may write down explicitly the induced actions $\phi_i : Q_i \to \text{Aut}(\mathbb{Z}^2)$ ($1 \leq i \leq 12$). Namely, for $j = 1, 2$,
\[ \phi_i(\alpha) : \lambda(a_j) \mapsto \mu_i(\alpha) \lambda(a_j) \mu_i(\alpha)^{-1}, \]
\[ \phi_i(\beta) : \lambda(a_j) \mapsto \mu_i(\beta) \lambda(a_j) \mu_i(\beta)^{-1}, \]
\[ \phi_i(t) : \lambda(a_j) \mapsto \lambda(t) \lambda(a_j) \lambda(t)^{-1} = A(a_j). \]
In fact, we see that for $q = \alpha$ or $\beta$,
\[
\phi_i(q) = P^{-1}[\mu_i(q)_{2 \times 2}] P,
\]
where $[\mu_i(q)_{2 \times 2}]$ is the matrix given by first two rows and two columns of $\mu_i(q)$. By Remark 5.4, we can use all the possible diagonalizing matrices $P$ of $A$ and all weakly conjugate matrices to $A$, namely,
\[
P = CP_0R \quad \text{(when $\psi_i(q) = 1$),} \quad P = CE P_0R \quad \text{(when $\psi_i(q) = -1$),}
\]
where $C = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$, $E = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $R \in \text{GL}(2, \mathbb{Z})$.

We want to find $i$'s for which $\phi_i(q) = P^{-1}[\mu_i(q)_{2 \times 2}] P \in \text{GL}(2, \mathbb{Z})$ for some $P$. Since $R \in \text{GL}(2, \mathbb{Z})$, in the following discussion we may assume $R = I$.

**Remark 5.6 (Case 1 of Remark 5.5.)** This occurs when $Q = Q_1, Q_4, Q_5, Q_6, Q_7, Q_8$ or $Q_{12}$. By Remark 5.5, $\phi(q) = P^{-1}[\mu(q)_{2 \times 2}] P = \pm I$ and so $[\mu(q)_{2 \times 2}] = \pm I$. However, when $Q$ is $Q_1$ with $\Phi = (x^2)$, $Q_4$, $Q_7$ or $Q_{12}$, by Remark 5.4, we see that $[\mu(q)_{2 \times 2}] \neq \pm I$. Hence these cases are not qualified. For the remaining $Q$'s, from Remark 5.4 we see that $[\mu(q)_{2 \times 2}] = -I$ and hence $\phi(q) = -I$. Explicitly, we must have
\[
\phi_1(\beta) = -I \quad \text{when} \quad Q = Q_1 \quad \text{with} \quad \Phi = (x^2),
\]
\[
\phi_5(\alpha) = -I \quad \text{when} \quad Q = Q_5,
\]
\[
\phi_6(\beta) = -I \quad \text{when} \quad Q = Q_6,
\]
\[
\phi_9(\alpha) = -I \quad \text{when} \quad Q = Q_8.
\]

**Remark 5.7 (Case 3 of Remark 5.5.)** This occurs when $Q = Q_3, Q_8$ or $Q_9$, in which $\phi_i(q) = M$ must be of type $U_2$ satisfying $M^2 = I$, $\det M = -1$ and $\det M = A^{-1}$. By Remark 5.5,
\[
[\mu_i(q)_{2 \times 2}] = \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

We want to know whether
\[
\pm P^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} P = M \in \text{GL}(2, \mathbb{Z})
\]
for some diagonalizing matrix $P = CE P_0$ of $A$ (noting that in these cases, $\psi_i(q) = -1$). If such an $M$ exists, it must be traceless. With $w = d/c$, we observe that
\[
\pm P^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} P = \pm P_0^{-1} \begin{bmatrix} w & 0 \\ 1 & w \end{bmatrix} P_0 = M.
\]

Moreover,
\[
P_0 M P_0^{-1} = \pm \begin{bmatrix} w & 0 \\ 1 & w \end{bmatrix} \Rightarrow P_0 (\det M) P_0^{-1} = P_0 A^{-1} P_0^{-1} \Rightarrow \det M = A^{-1}.
\]

Conversely, if $M \in \text{GL}(2, \mathbb{Z})$ is traceless and of determinant $-1$ satisfying $\det M = A^{-1}$, then we shall show that we can choose a nonzero real number $w$ such that the identity (5.2) holds. Since $\det M = -1$, we have $m_{11}^2 + m_{12} m_{21} = 1$. The identity $\det M = A^{-1}$ induces the identity
\[
(\ell_{11} - \ell_{22}) m_{11} + \ell_{12} m_{21} + \ell_{21} m_{12} = 0 \quad \text{(noting $m_{22} = -m_{11}$)}.
\]

Take
\[
w = \pm \left( -m_{11} + \sqrt{\ell_{11}^2 - \ell_{22}^2 + (\ell_{11} + \ell_{22})^2 - 4 m_{12}^2} \frac{m_{21}}{m_{12}} \right).
\]
By the above observation, we see that
\[
\frac{1}{w} = \pm \left( -m_{11} + \frac{(\ell_{11} - \ell_{22}) - \sqrt{(\ell_{11} + \ell_{22})^2 - 4m_{21}}}{2\ell_{21}} \right).
\]

With this choice of \( w \) or with the choice of \( C = \begin{bmatrix} w & 0 \\ 0 & 1 \end{bmatrix} \), it is now easy to check that
\[
P = CEP_0 = \begin{bmatrix} w & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} wy_1 & wy_2 \\ x_1 & x_2 \end{bmatrix},
\]
\[
P^{-1} \begin{bmatrix} e^{\theta_0} & 0 \\ 0 & e^{-\theta_0} \end{bmatrix} P = A, \quad \pm P^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} P = M.
\]

By Remark 3.5, the first two identities imply that we may realize our lattice \( \Gamma_A \)
\[
\langle a_1, a_2, t \mid [a_1, a_2] = 1, \quad ta/t^{-1} = A(a_i), \quad i = 1, 2 \rangle
\]
in \( \text{Aff}(\mathbb{R}^3) \) by taking
\[
\lambda(a_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \lambda(a_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \lambda(t) = \begin{bmatrix} e^{\theta_0} & 0 & 0 \\ 0 & e^{-\theta_0} & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

The last identity implies that
\[
\lambda(\beta) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]
acts on the subspace \( \langle \lambda(a_1), \lambda(a_2) \rangle \) via \( M \), namely, \( \lambda(\beta)\lambda(a_1)\lambda(\beta)^{-1} = M(\lambda(a_1)) \). Furthermore, we see that
\[
\lambda(\beta)^2 = I \quad \text{and} \quad \lambda(\beta)\lambda(t)\lambda(\beta)^{-1} = \lambda(t)^{-1}.
\]
Consequently, among all the imbeddings of \( \Gamma_A \subset \text{Sol} \) into \( \text{Aff}(\mathbb{R}^3) \), the above imbedding fits in the diagram (4.1) when \( Q = \{ t, \beta \mid \beta^2 = 1, \beta t \beta^{-1} = t^{-1} \} \).

We also remark the following: Suppose \( B \) is weakly conjugate to \( A \). So, \( B = G^{-1}A^{\pm 1}G \) for some \( G \in \text{GL}(2, \mathbb{Z}) \). Then \( MAM^{-1} = A^{-1} \) is equivalent to \( (G^{-1}MG)B(G^{-1}MG)^{-1} = B^{-1} \) and \( \text{tr}(G^{-1}MG) = \text{tr}M \) and \( \det(G^{-1}MG) = \det M \).

We are interested in finding which \( A \in \text{SL}(2, \mathbb{Z}) \) with trace > 2 has a solution \( M \) satisfying that \( M \) is traceless, \( \det M = -1 \) (note that these two conditions on \( M \) induce \( M^2 = I \)) and \( MAM^{-1} = A^{-1} \). This problem is equivalent to find the existence of an integer solution \( (x, y, z) \) for the system
\[
\begin{cases}
x^2 + yz = 1, \\
(\ell_{11} - \ell_{22})x + \ell_{12}y + \ell_{21}z = 0.
\end{cases}
\]

To the best of our knowledge, this problem has not been solved in general.

Since \( M \) is traceless of determinant \(-1\), \( I + M \) is a non-zero singular matrix and so \( \ker(I + M) \) is 1-dimensional. We can choose a generator \( [r_1, r_2]^\top \) of \( \ker(I + M) \): note that \( \text{gcd}(r_1, r_2) = 1 \) (See Lemma 7.2). Now, choose \( s_1, s_2 \) so that \( r_1 s_2 - r_2 s_1 = 1 \). Let
\[
Q = \begin{bmatrix} r_1 & s_1 \\ r_2 & s_2 \end{bmatrix} \in \text{SL}(2, \mathbb{Z}), \quad M' = Q^{-1}MQ = \begin{bmatrix} -1 & m_{12}' \\ 0 & 1 \end{bmatrix}, \quad A' = Q^{-1}AQ.
\]

Further, we take
\[
Q' = \begin{bmatrix} 1 & m_{12}' \\ 0 & 2 \end{bmatrix} \quad \text{(when } m_{12}' \text{ is even)}; \quad Q' = \begin{bmatrix} 1 & m_{12}' - 1 \\ 0 & 2 \end{bmatrix} \quad \text{(when } m_{12}' \text{ is odd)}.
\]
Then we can check easily that
\[ M'' = Q^{-1}M'Q' = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ (when } m_{12} \text{ is even);} \]
\[ M''' = Q^{-1}M'Q' = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \text{ (when } m_{12} \text{ is odd).} \]

Let \( A' := Q^{-1}A'Q' \). Since \( M''A''M''^{-1} = A''^{-1} \), it follows that when \( m_{21}' \) is even, \( \ell''_{11} = \ell''_{22} \), and when \( m_{21}' \) is odd, \( \ell''_{21} = \ell''_{11} - \ell''_{22} \). Since \( A \) and \( A' \) are (weakly) conjugate to each other, they may determine the same lattice of \( \text{Sol} \). Therefore, we may assume that \( M \) satisfies \( m_{11} = -1 \) and \( m_{21} = 0 \) and \( m_{12} = \frac{\ell_{11}-\ell_{22}}{\ell_{21}} = 0 \) or 1. Note further that if \( m_{12} = \frac{\ell_{11}-\ell_{22}}{\ell_{21}} = 1 \) then \( \ell_{21} = \ell_{11} - \ell_{22} \neq 0 \) and so det \( A = \ell_{11}\ell_{22} - (\ell_{11} - \ell_{22})\ell_{12} = 1 \Rightarrow \ell_{12} = \frac{\ell_{11}\ell_{22}-1}{\ell_{11} - \ell_{22}} \).

In the above, when \( m_{12}' \) is odd, instead of taking \( Q' = \begin{bmatrix} 1 & m_{12}'^{-1} \\ 0 & 1 \end{bmatrix} \), we may take \( Q' = \begin{bmatrix} m_{12}'^{-1} & m_{12}'^{-1} \\ 0 & 1 \end{bmatrix} \). This new choice results in \( M'' = Q^{-1}M'Q' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \). The condition \( MAM^{-1} = A^{-1} \) is equivalent to the condition \( \ell_{12} = -\ell_{21} \).

**Remark 5.8** (Case 4 of Remark 5.5.) This occurs when \( Q = Q_{10} \) or \( Q_{11} \), in which \( \phi(q) = M \) must be of type \( U_2 \) satisfying \( M^4 = I \) and \( MAM^{-1} = A^{-1} \). By Remark 5.5,
\[
\begin{bmatrix} \mu_i(q)_{2 \times 2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\]

We want to know whether
\[
P^{-1} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} P = M(= \phi(q)) \in \text{GL}(2, \mathbb{Z})
\]
for some diagonalizing matrix \( P = CEP_0 \) of \( A \) (noting that in these cases, \( \psi(q) = -1 \)). If such an \( M \) exists, it must be traceless and of determinant 1. With \( w = -d/c \), we observe that
\[
P^{-1} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} P = P_0^{-1} \begin{bmatrix} 0 & w \\ -1 & 0 \end{bmatrix} P_0 = M. \tag{5.4}
\]

Moreover, the above assumption yields that
\[
P_0MP_0^{-1} = \begin{bmatrix} 0 & w \\ -1 & 0 \end{bmatrix},
\]
\[
P_0(MAM^{-1})P_0^{-1} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} e^{\theta_0} & 0 \\ 0 & e^{-\theta_0} \end{bmatrix} \begin{bmatrix} 0 & w \\ -1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & w \\ e^{\theta_0} & e^{-\theta_0} \end{bmatrix}^{-1} = P_0A^{-1}P_0^{-1}.
\]

Hence we have \( MAM^{-1} = A^{-1} \), which is equivalent to the identity
\[
(\ell_{11} - \ell_{22})m_{11} + \ell_{12}m_{21} + \ell_{21}m_{12} = 0 \quad \text{ (noting } m_{22} = -m_{11}). \tag{5.5}
\]

Assume on the contrary that there is a traceless \( M \in \text{SL}(2, \mathbb{Z}) \) satisfying \( MAM^{-1} = A^{-1} \). Since \( \det M = 1 \), we have \( m_{11}^2 + m_{21}m_{12} + 1 = 0 \). From the condition \( MAM^{-1} = A^{-1} \), we have (5.5):
\[
(\ell_{11} - \ell_{22})m_{11} + \ell_{12}m_{21} + \ell_{21}m_{12} = 0.
\]

Take
\[
w = -m_{11} + (\ell_{11} - \ell_{22}) + \sqrt{(\ell_{11} + \ell_{22})^2 - 4 \ell_{21}},
\]

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By the above observation, we see that
\[ \frac{1}{w} = m_{11} - \frac{(\ell_{11} - \ell_{22}) - \sqrt{(\ell_{11} + \ell_{22})^2 - 4}}{2\ell_{21}}. \]

With this choice of \( w \) or with the choice of \( C = \begin{bmatrix} w & 0 \\ 0 & -1 \end{bmatrix} \), it is now easy to check that
\[ P = CE \begin{bmatrix} w & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} wy_1 & wy_2 \\ -x_1 & -x_2 \end{bmatrix}, \]
\[ P^{-1} \begin{bmatrix} e^b & 0 \\ 0 & e^{-b} \end{bmatrix} P = A, \quad P^{-1} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} P = M. \]

By Remark 3.5, the first two identities imply that we may realize our lattice \( \Gamma_A \)
\[ \{a_1, a_2, t \mid [a_1, a_2] = 1, \ t a_i t^{-1} = A(a_i), \ i = 1, 2 \} \]
in \( \text{Aff}(\mathbb{R}^3) \) by taking
\[ \lambda(a_1) = \begin{bmatrix} 1 & 0 & 0 & wy_1 \\ 0 & 1 & 0 & -x_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \lambda(a_2) = \begin{bmatrix} 1 & 0 & 0 & wy_2 \\ 0 & 1 & 0 & -x_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \lambda(t) = \begin{bmatrix} e^b & 0 & 0 & 0 \\ 0 & e^{-b} & 0 & 0 \\ 0 & 0 & 1 & t_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \]

The last identity implies that
\[ \lambda(\alpha) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]
acts on the subspace \( (\lambda(a_1), \lambda(a_2)) \) via \( M \), namely, \( \lambda(\alpha)\lambda(a_i)\lambda(\alpha)^{-1} = M(\lambda(a_i)) \). Furthermore, we see that
\[ \lambda(\alpha)^4 = I \]
and \( \lambda(\alpha)\lambda(t)\lambda(\alpha)^{-1} = \lambda(t)^{-1} \). Consequently, among all the imbeddings of \( \Gamma_A \subset \text{Sol} \) into \( \text{Aff}(\mathbb{R}^3) \), the above imbedding fits in the diagram (4.1) when \( Q = \{ t, \alpha \mid \alpha^4 = 1, \alpha t \alpha^{-1} = t^{-1} \} \).

Similar to the previous, we are interested in finding which \( A \in \text{SL}(2, \mathbb{Z}) \) with trace \( > 2 \) has a solution \( M \) satisfying that \( M \) is traceless, \( \det M = 1 \) (hence \( M^2 = -I \)) and \( MAM^{-1} = A^{-1} \). This problem is equivalent to find the existence of an integer solution \( (x, y, z) \) for the system
\[ \begin{cases} x^2 + yz + 1 = 0, \\ (\ell_{11} - \ell_{22})x + \ell_{12}y + \ell_{21}z = 0. \end{cases} \]

As before, we do not know how to solve this problem in general. We can solve in special cases.

For example, if \( A \) is symmetric, the system has integer solutions \( (x, y, z) = \pm(0, 1, -1) \). This implies that we can choose a basis of the lattice \( \mathbb{Z}^2 = \Gamma_A \cap \mathbb{R}^2 \) with respect to which \( q \in Q \) acts on the lattice as
\[ \phi(q) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \]

We consider a non-symmetric \( A \):
\[ A = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} \quad (\det A = 1, \text{tr} A = 3 > 2, \text{not symmetric}). \]

The system has integer solutions \( (x, y, z) = \pm(1, 2, -1) \). Next we consider another non-symmetric \( A \):
\[ A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \quad (\ell_{11} = \ell_{22}; \ell_{12}, \ell_{21} \mid (\ell_{11} - \ell_{22})). \]

The system becomes \( x^2 + 3y + z = 0 \). These induce \( x^2 + 1 = 3y^2 \). Up to modulo 4, the left-hand side is 1 or 2, but the right-hand side is 0 or 3. Hence there is no integer solution.

Using Remarks 5.4–5.8, we have the following \( \phi : Q \to \text{Aut}(\mathbb{Z}^2) \):
Lemma 5.9 Given a lattice $\Gamma_\lambda$ of Sol, $\Phi \subset D(4)$ and $\psi : \Phi \subset D(4) \to \text{Aut}(\mathbb{Z})$, only the following $Q$ may admit actions $\phi : Q \to \text{Aut}(\mathbb{Z}^2)$ compatible with $\psi$ and $\Gamma_\lambda$:

1. $\Phi \cong \mathbb{Z}_2 = \langle \beta \rangle$, $\psi(\beta) = 1$,
   - $Q_1 = \{ t, \beta \mid \beta^2 = 1, \beta t \beta^{-1} = t \} = \mathbb{Z} \times \mathbb{Z}_2$,
   - $\mu(\beta) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,
   - $\phi(\beta) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.
2. $\Phi \cong \mathbb{Z}_2 = \langle \beta \rangle$, $\psi(\beta) = 1$,
   - $Q_2 = \{ t, \beta \mid \beta^2 = t, \beta t \beta^{-1} = t \} = \mathbb{Z}$,
   - $\mu(\beta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ e^{\theta/2} & 0 & 0 & 0 \\ 0 & e^{-\theta/2} & 0 & 0 \\ 0 & 0 & 1 & t_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,
   - $\phi(\beta) = -\begin{bmatrix} \ell_{11} - 1 \\ \ell_{12} \\ \ell_{21} \\ \ell_{22} - 1 \\ \ell_{11} + \ell_{22} - 2 \\ \ell_{11} + \ell_{22} - 2 \end{bmatrix} \in \text{GL}(2, \mathbb{Z})$,
   - $\mu(\beta) = \begin{bmatrix} e^{\theta/2} & 0 & 0 & 0 \\ 0 & e^{-\theta/2} & 0 & 0 \\ 0 & 0 & 1 & t_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,
   - $\phi(\beta) = -\begin{bmatrix} \ell_{11} - 1 \\ \ell_{12} \\ \ell_{21} \\ \ell_{22} - 1 \\ \ell_{11} + \ell_{22} - 2 \\ \ell_{11} + \ell_{22} - 2 \end{bmatrix} \in \text{GL}(2, \mathbb{Z})$,
3. $\Phi \cong \mathbb{Z}_2 = \langle \beta \rangle$, $\psi(\beta) = -1$,
   - $Q_3 = \{ t, \beta \mid \beta^2 = 1, \beta t \beta^{-1} = t^{-1} \} = \mathbb{Z} \times \mathbb{Z}_2$,
\[ \phi(\beta) = M \in \text{GL}(2, \mathbb{Z}) \text{ where } \det M = -1, \text{tr}M = 0 \text{ and } MAM^{-1} = A^{-1}, \]

\[ \phi(\beta) = M \in \text{GL}(2, \mathbb{Z}) \text{ where } \det M = -1, \text{tr}M = 0 \text{ and } MAM^{-1} = A^{-1}; \]

\( Q_5 = \{ t, \alpha, \beta \mid \alpha^2 = 1, \beta^2 = t, at\alpha^{-1} = t, \beta \beta^{-1} = t^{-1}, [\alpha, \beta] = 1 \} = \langle \beta \rangle \times \langle \alpha \rangle = \mathbb{Z} \times \mathbb{Z}_2, \)

\[ \mu(\alpha) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \]

\[ \phi(\alpha) = \begin{bmatrix} e^{\phi/2} & 0 & 0 & 0 \\ 0 & e^{-\phi/2} & 0 & 0 \\ 0 & 0 & 1 & \sqrt{\ell_{12} - 1} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \]

\[ \phi(\beta) = \begin{bmatrix} \ell_{11} - 1/2 & \ell_{12} \sqrt{\ell_{11} + \ell_{22} - 2} \\ \ell_{21} \sqrt{\ell_{11} + \ell_{22} - 2} & \ell_{22} - 1/2 \end{bmatrix} \in \text{GL}(2, \mathbb{Z}); \]

\( Q_6 = \{ t, \alpha, \beta \mid \alpha^2 = 1, \beta^2 = 1, at\alpha^{-1} = t, \beta \beta^{-1} = t^{-1}, [\alpha, \beta] = 1 \} = \langle t \rangle \times \langle \beta \rangle = \langle \alpha \rangle \times \langle \beta \rangle = \mathbb{Z} \times \mathbb{Z}_2, \)

\[ \mu(\alpha) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \]

\[ \phi(\alpha) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \]

\[ \mu(\beta) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \]

\[ \phi(\beta) = M \in \text{GL}(2, \mathbb{Z}) \text{ where } \det M = -1, \text{tr}M = 0 \text{ and } MAM^{-1} = A^{-1}; \]

\( Q_7 = \{ t, \alpha, \beta \mid \alpha^2 = t, \beta^2 = 1, at\alpha^{-1} = t, \beta \beta^{-1} = t^{-1}, [\alpha, \beta] = 1 \} = \langle \alpha \rangle \times \langle \beta \rangle = \mathbb{Z} \times \mathbb{Z}_2, \)
\[ \mu(\alpha) = \begin{bmatrix}
e^{b_i/2} & 0 & 0 & 0 \\
e^{-b_i/2} & 0 & 0 & 0 \\
0 & 0 & 1 & \frac{t_0}{2} \\
0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{bmatrix}, \]

\[ \phi(\alpha) = -\begin{bmatrix}
\ell_{11} + 1 & \ell_{12} \\
\ell_{21} & \ell_{22} + 2 \end{bmatrix} \in \text{GL}(2, \mathbb{Z}), \]

\[ \mu(\beta) = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \end{bmatrix}. \]

(7) \( \Phi \cong \mathbb{Z}_4 = \langle \alpha \rangle, \psi(\alpha) = -1, \)

(8) \( \Phi = D(4) = \langle \alpha, \beta \rangle, \psi(\alpha) = -1, \psi(\beta) = 1, \)

\[ \phi(\alpha) = M \in \text{GL}(2, \mathbb{Z}) \text{ where } \det M = -1, \text{tr} M = 0 \text{ and } MAM^{-1} = A^{-1}; \]

\[ \phi(\beta) = M \in \text{GL}(2, \mathbb{Z}) \text{ where } \det M = 1, \text{tr} M = 0 \text{ and } MAM^{-1} = A^{-1}; \]

\( Q_{10} = \{ t, \alpha \mid \alpha^4 = 1, \alpha t \alpha^{-1} = t^{-1} \} = \langle t \rangle \times \langle \alpha \rangle = \mathbb{Z} \times \mathbb{Z}_4, \)

\( Q_{11} = \{ t, \alpha, \beta \mid \alpha^4 = 1, \beta^2 = t, \alpha t \alpha^{-1} = t^{-1}, \beta t \beta^{-1} = t, \beta \alpha \beta^{-1} = t \alpha^{-1} \} = (\langle t \rangle \times \langle \alpha \rangle) \times \langle \beta \alpha \rangle = (\mathbb{Z} \times \mathbb{Z}_4) \times \mathbb{Z}_2, \)

\( \mu_{11}(\alpha) = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \end{bmatrix}, \)

\[ \phi(\alpha) = M \in \text{GL}(2, \mathbb{Z}) \text{ where } \det M = 1, \text{tr} M = 0 \text{ and } MAM^{-1} = A^{-1}, \]

\[ \mu_{11}(\beta) = \begin{bmatrix}
e^{b_i/2} & 0 & 0 & 0 \\
e^{-b_i/2} & 0 & 0 & 0 \\
0 & 0 & 1 & \frac{t_0}{2} \\
0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{bmatrix}, \]

\[ \phi(\beta) = -\begin{bmatrix}
\ell_{11} - 1 & \ell_{12} \\
\ell_{21} & \ell_{22} - 2 \end{bmatrix} \in \text{GL}(2, \mathbb{Z}). \]

**Proof.** Consider \( Q_1. \) Lemma 5.3 shows that we have either \( \beta \mapsto y, x^2y \) or \( x^2. \) By Remark 5.4, we then have respectively either

\[ \mu_i(\beta) = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{bmatrix}. \]
Thus by Remark 5.6

\[
\phi_1(\beta) = \pm P \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} P^{-1} = \pm \begin{bmatrix} \ell_{22} - \ell_{11} & -2\ell_{12} \\ \sqrt{(\ell_{11} + \ell_{22})^2 - 4} & \sqrt{(\ell_{11} + \ell_{22})^2 - 4} \end{bmatrix},
\]

or \( P \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} P^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \).

However, as \( \phi_1(\beta) \) must belong to \( \text{GL}(2, \mathbb{Z}) \), the first two cases are disqualified. Therefore we have

\[
\mu_1(\beta) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

and \( \phi_1(\beta) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \).

The argument is identical for the rest \( Q_i \)'s.

\[ \square \]

6 Special 2-cocycles of \( Q \) in \( \mathbb{Z}^2 \)

We are given groups \( Q \) with presentation of the form (5.1). In view of computing \( H^2_\phi(Q; \mathbb{Z}^2) \) in practice, let us consider an extension

\[ 0 \rightarrow \mathbb{Z}^2 \rightarrow E \rightarrow Q \rightarrow 1 \]

compatible with \( \phi \) and \( \psi \) in Lemma 5.9. Then \( E \) has a presentation of the form

\[
E = \left\langle a_1, a_2, t, a_1, a_2 \left| \begin{array}{l}
[a_1, a_2] = 1, \\
ta_i t^{-1} = A(a_i) \quad (i = 1, 2), \\
\phi(a_i) a_i \phi(a_i)^{-1} = \psi(a_i) (i, j = 1, 2), \\
a_i a_j a_i^{-1} = a_i k_i \psi(a_i) t_i \quad (1 \leq i \leq p), \\
\alpha_i t a_i^{-1} = a_i k_i \psi(a_i) \quad (i = 1, 2) \end{array} \right. \right\rangle.
\] (6.1)

Without confusion again, we will abuse the symbol \( \alpha_i \) as elements of both \( E \) and \( Q \) when \( \alpha_i \in E \) is mapped to \( \alpha_i \in Q \) under the natural quotient map \( E \rightarrow Q \). Given \( Q \), this \( E \) is completely determined by the set \( K \) of integer vectors \( k = [k_{11}, k_{22}] \) and \( k' = [k_{11}', k_{22}'] \). But we cannot choose these integer vectors completely freely. We refer to \( E(K) \) as the group \( E \) determined by \( K \). A set \( K \) of integer vectors for which there exists a group \( E(K) \) as an extension of \( \mathbb{Z}^2 \) by \( Q \) is said to be computationally consistent.

In order to find all the possible inequivalent extensions \( E(K) \) of \( \mathbb{Z}^2 \) by \( Q \), we need to compute the cohomology-group \( H^2_\phi(Q, \mathbb{Z}^2) \). We should mention that computing the cohomology of groups such as these is an important problem in the cohomology of groups. For the computation in this section, we will use the methods employed in [3]. The extensions \( E(K) \) considered in [3] are virtually 2-nilpotent groups. Every such a group has a faithful representation into \( \text{Aff}(\mathbb{R}^3) \) (see [16]) and such a representation was used in [3]. However, our extensions \( E(K) \) are virtually 2-step solvable groups, and so we do not know whether our extensions \( E(K) \) have faithful representations into \( \text{Aff}(\mathbb{R}^3) \). Thus we need first to find out all the possible extensions and then we will find in the next section which of them has a faithful representation into \( \text{Aff}(\mathbb{R}^3) \).

The elements of \( E(K) \) can be written uniquely as words \( a_1^{p_1} a_2^{p_2} t^q u(a) \) \( (p_1, p_2, q \in \mathbb{Z}) \). Take a section \( s : Q \rightarrow E(K) \), \( q = t^q u(a) \mapsto a_1^{p_1} a_2^{p_2} t^q u(a) \), which we will refer to as the standard section.

**Definition 6.1** The cocycle \( f_K : Q \times Q \rightarrow \mathbb{Z}^2 \) determined by the standard section is called a special cocycle. The set of special cocycles \( \{ f_K \mid K \text{ computationally consistent} \} \) will be denoted by \( SZ_\phi^2(Q, \mathbb{Z}^2) \).

Since every extension of \( \mathbb{Z}^2 \) by \( Q \) is equivalent to some \( E(K) \), every cocycle in \( SZ_\phi^2(Q, \mathbb{Z}^2) \) is cohomologous to a special cocycle. So it will be sufficient to work with \( SZ_\phi^2(Q, \mathbb{Z}^2) \) to obtain the cohomology-group. The following proposition shows that computing \( H^2_\phi(Q, \mathbb{Z}^2) \) based on these special cocycles is possible. However, in
practice, it is very hard to compute $H^2_\phi(Q, \mathbb{Z}^2)$ explicitly in most cases as it depends on the defining matrix $A$ of the lattice $\Gamma_A$ of Sol. Compare our proposition with [3, Proposition 3.2]. As we can not assume any embedding our extensions $E(K)$ into $\text{Aff}(\mathbb{R}^3)$, our proposition is much general.

**Proposition 6.2**

1. $SZ^2_\phi(Q, \mathbb{Z}^2)$ is a subgroup of $Z^2_\phi(Q, \mathbb{Z}^2)$. Moreover, if $K_1$ and $K_2$ are computationally consistent then $f_{K_1+K_2} = f_{K_1} + f_{K_2}$.
2. A special cocycle $f_K$ is a coboundary if and only if $K$ allows an integer solution to a well determined finite set of matrix equations.

**Proof.** The relations in (6.1) involving integer vectors of $K$ are

$$w_i(\alpha_1, \alpha_2) = a_i^{1} a_i^{2} t_i^1, \quad \alpha_i t_i = a_i^{1} a_i^{2} \psi(\alpha_i)(t).$$

These give rise to a system of matrix equations, the variables being the vectors $k$ in $K$. It is not hard to see that this system of matrix equations has a solution $K$ if and only if this $K$ is computationally consistent. In fact, this is a practical way to determine all computationally consistent $K$’s. It is clear that if $K$ and $K'$ are solutions for the system of matrix equations then so is $K + K'$. Recall that the 2-cocycle $f_K$ is defined by

$$f_K(x, y) = s(x) \cdot s(y) \cdot s(xy)^{-1}$$

for all $x, y \in Q$, and we can show that $f_K$ satisfies

$$f_K(t^q, 1) = f_K(1, t^q) = f_K(t, t) = 0, \quad f_K(\alpha_i, t) = k_i, \quad f_K(\alpha_i, \alpha_j) = k_{ij}.$$ 

These together with the cocycle condition

$$xf_K(y, z) - f_K(xy, z) + f_K(x, yz) - f_K(x, y) = 0$$

determine $f_K$ completely. It follows now that $f_{K_1+K_2} = f_{K_1} + f_{K_2}$, which proves the first assertion.

For the second assertion, we assume

$$f_K(x, y) = \delta c(x, y) + c(x)$$

for some 1-cochain $c : Q \to \mathbb{Z}^2$, i.e., for some function $c$ with $c(1) = 0$. It is clear that

$$0 = f_K(t^q, \alpha_i) = A^q c(\alpha_i) - c(t^q \alpha_i) + c(t^q),$$

which implies that

$$c(t^q \alpha_i) = c(t^q) + A^q c(\alpha_i).$$

In a similar way, one verifies easily that

$$c(t^q) = \begin{cases} (I + A + A^2 + \cdots + A^{q-1})c(t) & \text{if } q > 0, \\ 0 & \text{if } q = 0, \\ -A^{-1}(I + A^{-1} + \cdots + A^{q+1})c(t) & \text{if } q < 0. \end{cases}$$

Thus, $c$ is completely determined by $c(t)$ and $c(\alpha_i)$.

Now, the problem of finding all $K$ such that $f_K$ is a coboundary is transformed to finding all $K$ for which the finite set of matrix equations in the variables $c(t)$ and $c(\alpha_i)$

$$f_K(t^q \alpha_i, t^q \alpha_j) = \delta c(t^q \alpha_i, t^q \alpha_j)$$

has an integer solution. \qed

**7 Extensions of $\mathbb{Z}^2$ by the groups $Q$**

In this section, based on Proposition 6.2 we will compute $H^2_\phi(Q; \mathbb{Z}^2)$ and write out the corresponding inequivalent extensions of $\mathbb{Z}^2$ by $Q$. Although this problem seems quite difficult to solve in general, it is often possible to
simplify a lot. We will work out explicitly only one case \( Q = Q_8 \), which is regarded as the most complicated one. The methods which are used in this case will give enough ideas for all the remaining cases. We can derive the results in a similar fashion and their details are left to the reader.

The following lemmas are useful in computing \( H^2_\phi(Q; \mathbb{Z}^2) \).

**Lemma 7.1** Every nonzero singular \( 2 \times 2 \) integer matrix can be written as

\[
\begin{bmatrix}
mk_2 & -mk_1 \\
nk_2 & -nk_1
\end{bmatrix}
\]

where \( m, n, k_1, k_2 \) are integers with \( \gcd(k_1, k_2) = 1 \).

**Proof.** Let

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]

be a nonzero singular integer matrix. Assume \( a \neq 0 \) or \( c \neq 0 \). Let \( h = \gcd(a, c) \). Then we have

\[
a = a_1 h, \quad c = c_1 h, \quad a_1 u + c_1 v = 1, \quad a_1 d - bc_1 = 0
\]

for some integers \( a_1, c_1 \) and \( u, v \). Let \( g' = \gcd(h, bu + dv) \). Then it can be shown easily that

\[
g' = \gcd(h, b, d) = \gcd(a, b, c, d) = g.
\]

\[
b = b(a_1 u + c_1 v) = a_1 bu + bc_1 v = a_1 bu + a_1 dv = a_1 (bu + dv),
\]

\[
d = d(a_1 u + c_1 v) = c_1 (bu + dv).
\]

Thus we can write

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix} = \begin{bmatrix}
a_1 g & h/g & -a_1 g \cdot (bu + dv) \\
c_1 g & h/g & -c_1 g \cdot (bu + dv)
\end{bmatrix}.
\]

If \( a = c = 0 \), then we can write

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix} = \begin{bmatrix}
-b \cdot 0 & b \cdot 1 \\
-d \cdot 0 & d \cdot 1
\end{bmatrix}.
\]

**Lemma 7.2** Let \( m, n, k_1, k_2 \) be integers with \( \gcd(k_1, k_2) = 1 \) and let

\[
K = \begin{bmatrix}
mk_2 & -mk_1 \\
nk_2 & -nk_1
\end{bmatrix}, \quad L = \pm \begin{bmatrix}
-nk_1 & mk_1 \\
-mk_2 & mk_2
\end{bmatrix}
\]

be nonzero integer matrices. Let

\[
g = \gcd(m, n), \quad u = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}, \quad v = \frac{1}{g} \begin{bmatrix} m \\ n \end{bmatrix}.
\]

Then we have

\[
\ker K = \mathbb{Z} u, \quad \im L = g \mathbb{Z} u, \quad \ker K/\im L \cong \mathbb{Z}_g,
\]

\[
\ker L = \mathbb{Z} v, \quad \im K = g \mathbb{Z} v, \quad \ker L/\im K \cong \mathbb{Z}_g.
\]

**Proof.** As \( \gcd(k_1, k_2) = 1 \), we have \( \ker K = \langle u \rangle = \mathbb{Z} u \). Since \( g = \gcd(m, n), m = gm_1, \ n = gn_1 \) for some integers \( m_1, n_1 \). Hence

\[
\im L = \left\{ \begin{bmatrix}
-nk_1 & mk_1 \\
-mk_2 & mk_2
\end{bmatrix} \\
\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} | b_1, b_2 \in \mathbb{Z} \right\} = \left\{ (mb_2 - nb_1)u \mid b_1, b_2 \in \mathbb{Z} \right\} = g \mathbb{Z} u.
\]
It follows that \( \ker K / \text{im} L = \mathbb{Z}u / g \mathbb{Z}u \cong \mathbb{Z}_g \). This proves the first half of our lemma, and the second half can be done in a similar way. \( \square \)

By Remark 5.6, any extension \( E \) of \( \mathbb{Z}^2 \) by \( Q_4, Q_6, Q_7 \) or \( Q_{12} \) cannot fit in the diagram (4.1). Hence we will consider the extensions of \( \mathbb{Z}^2 \) by the remaining \( Q \)'s.

### 7.1 Extensions by \( Q_1 \)

The extensions \( E_1 \) of \( \mathbb{Z}^2 \) by \( Q_1 \) have presentations of the form

\[
E_1(k, k') = \left\{ a_1, a_2, t, \beta \mid \begin{align*}
[ a_1, a_2 ] &= 1, \, ta_1t^{-1} = A(a_i) \quad (i = 1, 2), \\
\beta a_i \beta^{-1} &= a_i^{-1} \quad (i = 1, 2), \\
\beta^2 &= a_1^k a_2^k, \quad \beta t\beta^{-1} = a_1^{k_1} a_2^{k_2}t
\end{align*} \right\}.
\]

Then we have

- the computational consistency condition: \( k = 0 \).

For, with \( \phi(\beta) = -I = N \), we have, by Proposition 6.2,

\[
\begin{align*}
\beta^2 \beta &= \beta \beta^2 \Rightarrow a_1^k a_2^k \beta = \beta a_1^k a_2^k = N(a_1)^k N(a_2)^k \beta \Rightarrow k = NK \Rightarrow k = 0, \\
\beta^2 t \beta^{-2} &= \beta a_1^k a_2^k t \beta^{-1} \Rightarrow a_1^k a_2^k t a_1^{-k_1} a_2^{-k_2} = N(a_1)^k N(a_2)^k a_1^{k_1} a_2^{k_2}t \Rightarrow k - Ak = (I + N)k' \Rightarrow 0 = 0.
\end{align*}
\]

Moreover,

- the coboundary conditions: \( k = 0, \, k' = -2c(t) + (I - A)c(\beta) \) for some 1-cochain \( c : Q_1 \to \mathbb{Z}^2 \).

For, by Proposition 6.2, we have

\[
\begin{align*}
k &= f_k(\beta, \beta) = \delta c(\beta, \beta) = \beta \cdot c(\beta) - c(\beta^2) + c(\beta) = (N + I)c(\beta) = 0, \\
k' &= f_k(\beta, t) = \delta c(\beta, t) = \beta \cdot c(t) - c(\beta t) + c(\beta) = \beta \cdot c(t) - (c(t) + Ac(\beta)) + c(\beta) = (N - I)c(t) + (I - A)c(\beta) = -2c(t) + (I - A)c(\beta).
\end{align*}
\]

Hence all the inessential extensions are \( E_1(0, k') \) where \( k' \in H_\phi^2(Q_1; \mathbb{Z}^2) = \mathbb{Z}^2 / (2(\mathbb{Z}^2) + \text{im} (I - A)) \). Note also that \( H_\phi^2(Q_1; \mathbb{Z}^2) \) is a finite group and all the extensions are not torsion-free.

### 7.2 Extensions by \( Q_2 \)

Since \( Q_2 = \mathbb{Z} \) we have \( H_\phi^2(Q_2; \mathbb{Z}^2) = \{0\} \). Thus the extension \( E_2 \) is unique:

\[
E_2 = \{ a_1, a_2, \beta \mid [a_1, a_2] = 1, \beta a_i \beta^{-1} = \phi(\beta)(a_i) \}
\]

where \( \phi(\beta) \) is either, by Lemma 5.9,
\( \mu(\beta) = \begin{bmatrix} e^{h/2} & 0 & 0 & 0 \\ 0 & e^{-h/2} & 0 & 0 \\ 0 & 0 & 1 & t_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \)

\( \phi(\beta) = \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{11} & \ell_{12} \end{bmatrix} \)

\( \mu(\beta) = \begin{bmatrix} e^{h/2} & 0 & 0 & 0 \\ 0 & e^{-h/2} & 0 & 0 \\ 0 & 0 & 1 & t_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \)

\( \phi(\beta) = \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{11} & \ell_{12} \end{bmatrix} \)

\( \mu(\beta) = \begin{bmatrix} e^{h/2} & 0 & 0 & 0 \\ 0 & e^{-h/2} & 0 & 0 \\ 0 & 0 & 1 & t_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \)

\( \phi(\beta) = \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{11} & \ell_{12} \end{bmatrix} \)

By Case 2 of Remark 5.5, we are interested in \( A \) that has a square root. Assume \( A \) has a square root of the form (i) and hence also of the form (ii). It is now easy to see that taking \( \lambda(\beta) = \mu(\beta) \) we get two embeddings \( \lambda \) of \( E_2 \) into \( \text{Aff}(\mathbb{R}^3) \). These two embeddings are isomorphic by conjugation by an element

\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} \in \text{Aut}(\text{Sol}) \subset \text{Aff}(\text{Sol}) \subset \text{Aff}(\mathbb{R}^3).
\]

By Remark 5.5 again, if a square root \( \phi(\beta) \) of \( A \) has determinant \(-1\) then this is (ii); if \( \phi(\beta) \) has determinant 1, this is (iii). We pay special attention to the fact that if \( \phi(\beta) \) has determinant 1 then it has negative trace. Therefore \( \phi(\beta) \) cannot define a lattice of Sol, or \( E_2 \neq \Gamma_{\phi(\beta)} \).

### 7.3 Extensions by \( Q_3 \)

The extensions \( E_3 \) of \( \mathbb{Z}^2 \) by \( Q_3 \) have presentations of the form

\[
E_3(k, k') = \left\{ a_1, a_2, t, \beta \left| \begin{array}{l}
[a_1, a_2] = 1, \quad t a_i t^{-1} = A(a_i) \\
(i = 1, 2), \quad \beta a_i \beta^{-1} = \phi(\beta)(a_i) \\
(i = 1, 2), \quad \beta^2 = a_1^{k_1} a_2^{k_2}, \quad \beta t \beta^{-1} = a_1^{k_1} a_2^{k_2} t^{-1}
\end{array} \right. \right\},
\]

\( \phi(\beta) = M \in \text{GL}(2, \mathbb{Z}) \) where \( \det M = -1, \text{tr} M = 0 \) and \( M A M^{-1} = A^{-1} \).
By Remark 5.7, we may assume that \( m_{11} = -m_{22} = -1, m_{21} = 0 \) and \( m_{12} = \frac{\ell_{11} - \ell_{22}}{\ell_{21}} = 0 \) or 1. By Proposition 6.2, we have

- the computational consistency conditions: \( (M - I)k = 0, \) \( (I - A)k = (M - A)k' \),
- the coboundary conditions:

\[
k = (M + I)c(\beta), \quad k' = (M + A^{-1})c(t) + (I - A^{-1})c(\beta)
\]

for some 1-cochain \( c : Q_3 \to \mathbb{Z}^2 \).

Thus the computational consistency conditions become

\[
(I - M)k = 0, \quad (A - M)(k' - k) = 0
\]

with coboundary conditions

\[
k = (I + M)c(\beta), \quad k' - k = (A^{-1} + M)(c(t) - c(\beta)).
\]

Consequently,

\[
(k, k' - k) \in H^2_{\text{c}}(Q_3; \mathbb{Z}^2) \cong \frac{\ker(I - M)}{\im(I + M)} \oplus \frac{\ker(A - M)}{\im(A^{-1} + M)}.
\]

By Lemma 7.2, \( H^2_{\text{c}}(Q_3; \mathbb{Z}^2) \cong \mathbb{Z}_g \oplus \mathbb{Z}_h \) where \( g = \gcd(2, \ell_{11} - \ell_{22}) \) and \( h = \gcd(\ell_{11} + 1, \ell_{12} - \ell_{11} - \ell_{21}, \ell_{22} - 1) \). These information will give us all the inequivalent extensions \( E_3(k, k') \). In particular, the extensions corresponding to the choice \( k = 0 \) are not torsion-free, and the other extensions are torsion-free.

### 7.4 Extensions by \( Q_5 \)

The extensions \( E_5 \) of \( \mathbb{Z}^2 \) by \( Q_5 \) have presentations of the form

\[
E_5(k, k') = \left\{ a_1, a_2, \alpha, \beta \mid \begin{aligned}
[a_1, a_2] &= 1, \\
\alpha a_i a_i^{-1} &= a_i^{-1}, \\
\beta a_i a_i^{-1} &= \phi(\beta)(a_i) \\
\alpha^2 &= a_i a_i^{-1}, \quad \alpha \beta a^{-1} = a_i^{-1} a_i a_i^{-1}, \quad \beta a_i a_i^{-1} = a_i^{-1} a_i a_i^{-1}, \quad a_i^{-1} a_i a_i^{-1} = a_i^{-1} a_i a_i^{-1}
\end{aligned} \right\}
\]

\[
\phi(\beta) = \left[ \begin{array}{c}
\frac{\ell_{11} - 1}{\sqrt{\ell_{11} + \ell_{22} - 2}} \\
\frac{\ell_{11} + \ell_{22} - 2}{\ell_{21}} \\
\frac{\ell_{22} - 1}{\sqrt{\ell_{11} + \ell_{22} - 2}} \\
\frac{\ell_{11} + \ell_{22} - 2}{\sqrt{\ell_{11} + \ell_{22} - 2}}
\end{array} \right] = N.
\]

By Proposition 6.2,

- the computational consistency condition is \( k = 0 \).

To find the coboundary conditions, we note that every element of \( Q_5 \) can be written uniquely as \( \beta^p \) or \( \beta^p \alpha \). Hence the standard section \( s : Q_5 \to E_5 \) will be given as follows: \( \beta^p \mapsto a_i^0 a_i^0 \beta^p, \) \( \beta^p \alpha \mapsto a_i^0 a_i^0 \beta^p \alpha \). Then

\[
f_{k}(\alpha, \beta) = s(\alpha)s(\beta)s(\alpha \beta)^{-1} = s(\alpha)s(\beta)s(\beta \alpha)^{-1} = \alpha \beta(\alpha \beta)^{-1} = \alpha \beta(a^{-k} \alpha \beta)^{-1} = \alpha \beta^{-1} \alpha^{-1} a^{-k} = a^{-k}
\]

\[
\Rightarrow k' = \alpha \cdot c(\beta) - c(\alpha \beta) + c(\alpha) = c(\alpha) - c(\beta) - c(\beta \alpha),
\]

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\[ f_k(\beta, \alpha) = s(\beta)s(\alpha)s(\beta \alpha)^{-1} = \beta \alpha(\beta \alpha)^{-1} = a^0 \]
\[ \Rightarrow 0 = \beta \cdot c(\alpha) - c(\beta \alpha) + c(\beta) \]
\[ \Rightarrow c(\beta \alpha) = Nc(\alpha) + c(\beta). \]

Hence

- the coboundary condition: \( k' = -2c(\beta) + (I - N)c(\alpha) \) where \( c : Q_5 \to \mathbb{Z}^2 \) is an 1-cochain.

Consequently,

\[ k' \in H_2^2(\{0, k'\}) = \frac{\mathbb{Z}^2}{2(\mathbb{Z}^2) + \text{im}(I - N)}. \]

These information will give us all the inequivalent extensions \( E_8(0, k') \). The cohomology group \( H_2^2(\{0, k'\}) \) is finite. All the extensions are not torsion-free.

### 7.5 Extensions by \( Q_8 \)

The extensions \( E_8 \) of \( \mathbb{Z}^2 \) by \( Q_8 \) have presentations of the form

\[ E_8(\mathbf{m}, \mathbf{m}', \mathbf{k}, \mathbf{k}', \mathbf{n}) = \left\{ \begin{array}{l}
[a_1, a_2] = 1, \tau a_i t^{-1} = A(a_i) \\
[\alpha a_i, a_i^{-1}] = a_i^{-1} \\
[\beta a_i, \beta^{-1}] = \phi(a_i) \\
\alpha a_i = a_i^1 a_i^2 a_i^3, \alpha \tau a_i^{-1} = a_i^1 a_i^2 a_i^3 \\
\beta a_i = a_i^1 a_i^2 a_i^3, \beta \tau a_i^{-1} = a_i^1 a_i^2 a_i^3 \\
[\alpha, \beta] = a_i^1 a_i^2 a_i^3.
\end{array} \right. \]

\[ \phi(\beta) = M \in \text{GL}(2, \mathbb{Z}) \quad \text{where} \quad \det M = -1, \text{tr} M = 0 \quad \text{and} \quad MAM^{-1} = A^{-1}. \]

By Remark 5.7, we may assume that \( m_{11} = -m_{22} = 1, m_{21} = 0 \) and \( m_{12} = \frac{C(I - M)}{e_{21}} = 0 \) or 1. By Proposition 6.2, we have

- the computational consistency conditions

\[ \mathbf{m} = 0, \quad (I - M)\mathbf{k} = \mathbf{0}, \quad (I - A)\mathbf{k} = (M - A)\mathbf{k}', \quad 2\mathbf{k} = -(I + M)\mathbf{n}, \quad (A^{-1} - I)\mathbf{n} = (A^{-1} + M)\mathbf{m}' + 2\mathbf{k}'; \]

- the coboundary conditions

\[ \mathbf{m} = 0, \quad \mathbf{m}' = -2c(t) + (I - A)c(\alpha), \quad \mathbf{k} = (M + I)c(\beta), \quad \mathbf{k}' = (M + A^{-1})c(t) + (I - A^{-1})c(\beta), \quad \mathbf{n} = (I - M)c(\alpha) - 2c(\beta), \quad c(\alpha \beta) = c(\alpha) - c(\beta) \]

for some 1-cochain \( c : Q_8 \to \mathbb{Z}^2 \).

It is not hard to see that the above computational consistency conditions yield the following system of matrix equations

\[ (I - M)\mathbf{k} = \mathbf{0}, \]
\[ (I + M)(\mathbf{n} + \mathbf{k}) = \mathbf{0}, \]
\[ (A - M)(\mathbf{k}' - \mathbf{k}) = \mathbf{0} \Leftrightarrow A(I - A^{-1}M)(\mathbf{k}' - \mathbf{k}) = \mathbf{0} \Leftrightarrow (I - A^{-1}M)(\mathbf{k}' - \mathbf{k}) = \mathbf{0}, \]
\[ (A^{-1} + M)(\mathbf{m}' - \mathbf{n} + M(\mathbf{k}' - \mathbf{k})) = \mathbf{0}. \]
and the coboundary conditions yield the following system of matrix equations

\[ \mathbf{k} = (I + M)\mathbf{c}(\beta), \]
\[ \mathbf{n} + \mathbf{k} = (I - M)(\mathbf{c}(\alpha) - \mathbf{c}(\beta)), \]
\[ \mathbf{k}^\prime - \mathbf{k} = (A^{-1} + M)(\mathbf{c}(t) - \mathbf{c}(\beta)), \]
\[ \mathbf{m}^\prime - \mathbf{n} + M(\mathbf{k}^\prime - \mathbf{k}) = (A - M)(-\mathbf{c}(\alpha) - M\mathbf{c}(\beta) + M\mathbf{c}(t)). \]

By letting

\[ \mathbf{a} = -\mathbf{c}(\alpha) - M\mathbf{c}(\beta) + M\mathbf{c}(t), \quad \mathbf{b} = \mathbf{c}(\beta), \quad \mathbf{c} = -M(\mathbf{c}(t) - \mathbf{c}(\beta)), \]

the coboundary conditions become

\[ \mathbf{k} = (I + M)\mathbf{b}, \]
\[ \mathbf{n} + \mathbf{k} = -(I - M)(\mathbf{a} + \mathbf{b} + \mathbf{c}), \]
\[ \mathbf{k}^\prime - \mathbf{k} = -(I + MA)\mathbf{c}, \]
\[ \mathbf{m}^\prime - \mathbf{n} + M(\mathbf{k}^\prime - \mathbf{k}) = (A - M)\mathbf{a}. \]

Let

\[ \mathcal{K} = \ker(I - M) \oplus \ker(I + M) \oplus \ker(I - A^{-1}M) \oplus \ker(A^{-1} + M), \]
\[ \mathcal{L} = \{((I + M)\mathbf{b}, -(I - M)(\mathbf{a} + \mathbf{b} + \mathbf{c}), -(I + MA)\mathbf{c}, (A - M)\mathbf{a}) \mid \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{Z}^2\}. \]

To evaluate \( H^1_\partial(Q_8; \mathbb{Z}^2) \cong \mathcal{K}/\mathcal{L} \), we shall use Lemma 7.2. Notice that

\[ I - M = \begin{bmatrix} 2 & -\ell_{11} - \ell_{22} \\ 0 & \ell_{21} \\ \ell_{21} & 0 \end{bmatrix}, \quad I + M = \begin{bmatrix} 0 & \ell_{11} - \ell_{22} \\ 0 & \ell_{21} \\ \ell_{21} & 2 \end{bmatrix}, \]
\[ I - A^{-1}M = \begin{bmatrix} 1 + \ell_{22} & \ell_{22}^2 - 1 \\ -\ell_{21} & 1 - \ell_{22} \end{bmatrix}, \quad I + MA = \begin{bmatrix} 1 - \ell_{22} & 1 - \ell_{22}^2 \\ \ell_{21} & \ell_{21} \\ \ell_{21} & \ell_{21} \end{bmatrix}, \]
\[ A^{-1} + M = \begin{bmatrix} \ell_{22} - 1 & (\ell_{11} + 1)(\ell_{22} - 1) \\ -\ell_{21} & \ell_{21} \end{bmatrix}, \quad A - M = \begin{bmatrix} \ell_{11} + 1 & (\ell_{11} + 1)(\ell_{22} - 1) \\ \ell_{21} & \ell_{21} \\ \ell_{21} & \ell_{22} - 1 \end{bmatrix}. \]

First we remark that the pairs \((K, L)\) of matrices where \((K, L) = (I - M, I + M), (I + M, I - M), (I - A^{-1}M, -(I + MA))\) and \((A^{-1} + M, A - M)\) satisfy the conditions of Lemma 7.2. Let

\[ g = \gcd\left(2, \frac{\ell_{11} - \ell_{22}}{\ell_{21}}\right), \]
\[ h = \gcd\left(1 \pm \ell_{22}, \frac{\ell_{22}^2 - 1}{\ell_{21}}, \ell_{21}\right), \]
\[ h' = \gcd(1 + \ell_{22}, \ell_{21}), \]
\[ k = \gcd(\ell_{11} + 1, \frac{(\ell_{11} + 1)(\ell_{22} - 1)}{\ell_{21}}, \ell_{21}, \ell_{22} - 1), \]
\[ k' = \gcd(\ell_{22} - 1, \ell_{21}). \]

Since \( A^{-1} + M = (I + MA)A^{-1} \), it follows that the gcd \( k \) of the entries of \( A^{-1} + M \) is a divisor of the gcd of the entries of \( I + MA \) and hence of \( I - A^{-1}M \), which is \( h \). Similarly, \( h \) divides \( k \) and hence \( h = k \). For some
integers \( p_1 \) and \( q_1 \), we have

\[
\frac{1 + \ell_{22}}{h'} p_1 + \frac{\ell_{21}}{h'} q_1 = 1, \quad \frac{\ell_{22} - 1}{k'} p_2 + \frac{\ell_{24}}{k'} q_2 = 1.
\]

By Lemma 7.2,

\[
\ker(I - M) = \langle e_1 \rangle = \left\{ \frac{1}{g} \left[ \begin{array}{c} \ell_{11} - \ell_{22} \\ \ell_{21} \\ 2 \end{array} \right] \right\},
\]

\[
\ker(I + M) = \langle e_2 \rangle = \left\{ \frac{1}{0} \right\},
\]

\[
\ker(I - A^{-1}M) = \langle e_3 \rangle = \left\{ \frac{1}{h} \left[ \begin{array}{c} \ell_{11} - \ell_{22} \\ \ell_{21} \\ h' \end{array} \right] \right\},
\]

\[
\ker(A^{-1} + M) = \langle e_4 \rangle = \left\{ \frac{1}{k} \left[ \begin{array}{c} \ell_{11} - 1 + (\ell_{22} - 1) \ell_{21} \\ \ell_{21} \\ k' \end{array} \right] \right\}
\]

and

\[
(I + M)b = gb_2 e_1,
\]

\[
(I - M)(a + b + c) = (2(a_1 + b_1 + c_1) - \ell_{11} - \ell_{22} (a_2 + b_2 + c_2)) e_2,
\]

\[
(I + M)\ell = h \left( \frac{\ell_{21}}{h'} c_1 + \frac{1 + \ell_{22}}{h'} c_2 \right) e_3,
\]

\[
(A - M)\ell = k \left( \frac{\ell_{21}}{k'} a_1 + \frac{\ell_{22} - 1}{k'} a_2 \right) e_4.
\]

These imply that \( K \) is isomorphic to the free Abelian group on four generators \( e_1, e_2, e_3 \) and \( e_4 \), and the subgroup \( \mathcal{L} \) of \( K \) is

\[
\left\{ gb_2 e_1 - (2(a_1 + b_1 + c_1) - \ell_{11} - \ell_{22} (a_2 + b_2 + c_2)) e_2 \\
- h \left( \frac{\ell_{21}}{h'} c_1 + \frac{1 + \ell_{22}}{h'} c_2 \right) e_3 + k \left( \frac{\ell_{21}}{k'} a_1 + \frac{\ell_{22} - 1}{k'} a_2 \right) e_4 \mid a_1, b_1, c_1 \in \mathbb{Z} \right\}
\]

\[
= \left\{ - a_1 \left( 2e_1 - k \frac{\ell_{21}}{k'} e_4 \right) + a_2 \left( \ell_{11} - \ell_{22} \frac{\ell_{21}}{\ell_{21}} e_2 + k \frac{\ell_{22} - 1}{k'} e_4 \right) - b_1 (2e_2) + b_2 \left( g e_1 + \frac{\ell_{11} - \ell_{22}}{\ell_{21}} e_2 \right) \\
- c_1 \left( 2e_2 + h \frac{\ell_{21}}{h'} e_3 \right) + c_2 \left( \ell_{11} - \ell_{22} \frac{\ell_{21}}{\ell_{21}} e_2 - h \frac{1 + \ell_{22}}{h'} e_3 \right) \mid a_1, b_1, c_1 \in \mathbb{Z} \right\}
\]

Hence \( \mathcal{L} \) is the subgroup of the free Abelian group \( K \) generated by the

\[
2e_1 - k \frac{\ell_{21}}{k'} e_4, \quad \ell_{11} - \ell_{22} \frac{\ell_{21}}{\ell_{21}} e_2 + k \frac{\ell_{22} - 1}{k'} e_4, \quad 2e_2, \quad g e_1 + \frac{\ell_{11} - \ell_{22}}{\ell_{21}} e_2,
\]

\[
2e_2 + h \frac{\ell_{21}}{h'} e_3, \quad \ell_{11} - \ell_{22} \frac{\ell_{21}}{\ell_{21}} e_2 - h \frac{1 + \ell_{22}}{h'} e_3.
\]
Consequently $\mathcal{K}/\mathcal{L}$ is isomorphic to the Abelian group

$$
\mathcal{K}/\mathcal{L} \cong \left\langle \begin{array}{c}
e_1, e_2, e_3, e_4 \mid 2e_1 - \frac{\ell_{21}}{k'} e_1 = 0, \frac{\ell_{11} - \ell_{22}}{\ell_{21}} e_2 + k \frac{\ell_{22} - 1}{k'} e_2 = 0, \\
2e_2 = 0,
2e_2 + h \frac{\ell_{21}}{h'} e_3 = 0, \frac{\ell_{11} - \ell_{22}}{\ell_{21}} e_2 - \frac{1 + \ell_{22}}{h'} e_3 = 0 \end{array} \right\rangle.
$$

Using the basis theorem for finitely generated Abelian groups, we can reduce the relation matrix of the above presentation to the matrix of the standard form. This determines $(k, n + k, k' - k, m' - n + M(k' - k)) \in H_2^\phi(Q_8; \mathbb{Z}^2) = \mathcal{K}/\mathcal{L}$, and hence all the inequivalent extensions $E_8(0, m', k, k', n)$. All of them are not torsion-free.

When we read off $m'$, $k$, $k$, $n$ from the cohomology group $H_2^\phi(Q_8; \mathbb{Z}^2)$, we have to notice that $k$ is a multiple of $e_1$, $n + k$ is a multiple of $e_2$, $k' - k$ is a multiple of $e_3$ and $m' - n + M(k' - k)$ is a multiple of $e_4$.

Now we will examine explicitly the relation matrix of this group which is

$$
\begin{bmatrix}
2 & 0 & 0 & \frac{-k}{k'} \\
0 & \frac{\ell_{11} - \ell_{22}}{\ell_{21}} & 0 & \frac{\ell_{22} - 1}{k'} \\
0 & 2 & 0 & 0 \\
0 & \frac{\ell_{11} - \ell_{22}}{\ell_{21}} & 0 & 0 \\
0 & 2 & \frac{h}{h'} & 0 \\
0 & \frac{\ell_{11} - \ell_{22}}{\ell_{21}} & -\frac{1 + \ell_{22}}{h'} & 0 \\
\end{bmatrix}
$$

If $g = 2$, i.e., if $\frac{\ell_{11} - \ell_{22}}{\ell_{21}} = 0$, then the relation matrix is reduced to

$$
\begin{bmatrix}
2 & 0 & 0 & \frac{-k}{k'} \\
0 & 0 & 0 & \frac{\ell_{22} - 1}{k'} \\
0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & \frac{h}{h'} & 0 & 0 \\
0 & -\frac{1 + \ell_{22}}{h'} & 0 & 0 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
2 & 0 & 0 & \frac{-k}{k'} \\
0 & 0 & 0 & \frac{\ell_{22} - 1}{k'} \\
0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & \frac{h}{h'} & 0 & 0 \\
0 & -\frac{1 + \ell_{22}}{h'} & 0 & 0 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
2 & 0 & 0 & \frac{-k}{k'} \\
0 & 0 & 0 & \frac{\ell_{22} - 1}{k'} \\
0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -\frac{1 + \ell_{22}}{h'} & 0 & 0 \\
\end{bmatrix}
$$

$$
\begin{bmatrix}
2 & 0 & 0 & \frac{-k}{k'} \\
0 & 0 & 0 & \frac{\ell_{22} - 1}{k'} \\
0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
$$
If \( g = 1 \), i.e., if \( \frac{\ell_1 - \ell_2}{\ell_{21}} = 1 \), then the relation matrix can be reduced as follows:

\[
\begin{bmatrix}
2 & 0 & 0 & -k\frac{\ell_{21}}{k'} \\
0 & 1 & 0 & k\frac{\ell_{22} - 1}{k'} \\
0 & 2 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 2 & h\frac{\ell_{21}}{h'} & 0 \\
0 & 1 & -h\frac{1 + \ell_{22}}{h'} & 0
\end{bmatrix} \rightarrow
\begin{bmatrix}
0 & 2 & 0 & k\frac{\ell_{21}}{k'} \\
0 & 1 & 0 & k\frac{\ell_{22} - 1}{k'} \\
0 & 2 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & h\frac{\ell_{21}}{h'} & 0 \\
0 & 1 & -h\frac{1 + \ell_{22}}{h'} & 0
\end{bmatrix} \rightarrow
\begin{bmatrix}
0 & 0 & 0 & k\frac{\ell_{21}}{k'} \\
0 & 1 & 0 & 0 \\
0 & 2 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & h\frac{\ell_{21}}{h'} & 0 \\
0 & 0 & h\frac{1 + \ell_{22}}{h'} & 0
\end{bmatrix}
\]

where \( E_1: \{\mathbf{e}_1', \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\} \) where \( \mathbf{e}_1' = \mathbf{e}_1 + \mathbf{e}_2 \), \( E_2 : \{\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3, \mathbf{e}_4\} \) where \( \mathbf{e}_2' = \mathbf{e}_2 + k\frac{\ell_{21}}{k'} \mathbf{e}_4 \) and \( K = \gcd\left(\frac{\ell_{21}}{k'}, 2\frac{\ell_{22} - 1}{k'}\right) = \gcd\left(\frac{\ell_{21}}{k'}, 2\right) \). If, in addition, \( \frac{\ell_{21}}{k'} \) is odd, i.e., \( K = 1 \), then

\[
\begin{bmatrix}
0 & 0 & 0 & kK \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & h\frac{\ell_{21}}{h'} & 0 \\
0 & 0 & h\frac{1 + \ell_{22}}{h'} & k\frac{\ell_{22} - 1}{k'}
\end{bmatrix}
\]

\[
\rightarrow
\begin{bmatrix}
0 & 0 & 0 & k \\\n0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & h\frac{\ell_{21}}{h'} & 0 \\
0 & 0 & h\frac{1 + \ell_{22}}{h'} & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 0 & 0 & k \\\n0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & h & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[;\]
if, in addition, \( \frac{\ell_{21}}{k} \) is even, i.e., \( K = 2 \), then \( \frac{\ell_{22}-1}{k} \) is odd and so

\[
\begin{bmatrix}
0 & 0 & 0 & 2k \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & h \frac{\ell_{21}}{h'} & 0 \\
0 & h \frac{1 + \ell_{22}}{h'} & k & \ell_{22} - 1 \\
0 & h & 0 & k \\
0 & 0 & 0 & k
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 0 & 0 & 2k \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & h \frac{\ell_{21}}{h'} & 0 \\
0 & h \frac{1 + \ell_{22}}{h'} & k \\
0 & 0 & 0 & k \\
0 & 0 & 0 & k
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 0 & 2h \frac{1 + \ell_{22}}{h'} & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & hH & 0 \\
0 & 0 & 0 & k
\end{bmatrix}
\]

where \( E_3 : \{e_1', e_2', e_3, e_4'\}, \quad e_4' = \frac{1+\ell_{22}}{h'} e_2 + e_4 \) and \( H = \gcd\left(2 \frac{1+\ell_{22}}{h'}, \frac{\ell_{21}}{h'}\right) = \gcd\left(2, \frac{\ell_{21}}{h'}\right) \).

Therefore,

\[
H^2_{\mathbb{Z}^2}(Q_8; \mathbb{Z}^2) \cong \mathcal{K}/\mathcal{E}
\cong \ker(I - M) \oplus \ker(I + M) \oplus \ker(I - A^{-1}M) \oplus \ker(A^{-1} + M)
\]

where

\[
\eta = \begin{cases}
2 & \text{when } \frac{\ell_{21}}{\gcd(\ell_{22}-1, \ell_{21})}, \frac{\ell_{22}}{\gcd(1+\ell_{22}, \ell_{21})} \text{ are even,} \\
1 & \text{otherwise.}
\end{cases}
\]

We shall give three examples which explain explicitly how to find \( H^2(Q_8; \mathbb{Z}^2) \) and from this how to obtain all the equivalent extensions \( E_8(0, m', k, k', n) \). In the following examples, using Remark 5.7, we may take another choice for \( A \) and \( M \) which is much simpler than the given choice. However, this new choice yields the isomorphic cohomology group \( H^2(Q_8; \mathbb{Z}^2) \) and the isomorphic extensions \( E_8 \). Hence we do not need to take a new choice.

**Example 7.3** Let \( A = \begin{bmatrix} 7 & -3 \\ -2 & 5 \end{bmatrix} \) and \( M = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \). Then \( \det M = -1, \ \tr M = 0, \ MAM^{-1} = A^{-1} \). Observe that

\[
\ker(I - M) = \langle e_1 \rangle = \left\langle \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\rangle,
\]

\[
(I + M) b = (b_1 - b_2) e_1,
\]

\[
\ker(I + M) = \langle e_2 \rangle = \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle.
\]
\[-(I - M) (a + b + c) = (a_1 - 3a_2) + (b_1 - 3b_2) + (c_1 - 3c_2) e_2,\]
\[\ker (I - A^{-1} M) = \langle e_1 \rangle = \begin{bmatrix} 5 \\ 3 \end{bmatrix},\]
\[-(I + MA) c = (3c_1 - 2c_2) e_3,\]
\[\ker (A^{-1} + M) = \langle e_4 \rangle = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.\]

Since
\[(b_1 - b_2) e_1 + ((a_1 - 3a_2) + (b_1 - 3b_2) + (c_1 - 3c_2)) e_2 + (3c_1 - 2c_2) e_3 + a_1 e_4\]
\[= a_1 (e_2 + e_4) - 3a_2 e_2 + b_1 (e_1 + e_2) - b_2 (e_1 + e_2) + c_1 (e_2 + 3e_3) - c_2 (3e_2 + 2e_3),\]
the Abelian group $H^2_\phi (Q_8; \mathbb{Z}^2)$ is isomorphic to
\[
\left\{ \begin{array}{c|ccc}
  e_1, e_2, e_3, e_4 & e_2 + e_4 = 0, & 3e_2 = 0, & e_1 + e_2 = 0, \\
  e_1 + 3e_2 = 0, & e_2 + 3e_3 = 0, & 3e_2 + 2e_3 = 0 \end{array} \right\}.
\]

Now, it is easy to see that the relation matrix is reduced to the trivial matrix. This means that
\[(k, n + k, k' - k, m' - n + M (k' - k)) \in H^2_\phi (Q_8; \mathbb{Z}^2) = \{0\}.
\]

Hence we may choose $k = n = k' = m' = 0$ and so there is a unique extension $E_8(0, 0, 0, 0, 0, 0)$.

**Example 7.4** Let $A = \begin{bmatrix} 1 & 4 \\ 1 & 4 \end{bmatrix}$ and $M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Then $det M = -1$, $tr M = 0$, $MAM^{-1} = A^{-1}$. We observe that
\[\ker (I - M) = \langle e_1 \rangle = \begin{bmatrix} 5 \\ 3 \end{bmatrix},\]
\[(I + M) b = (b_1 - b_2) e_1,\]
\[\ker (I + M) = \langle e_2 \rangle = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\]
\[-(I - M) (a + b + c) = (3a_1 - 5a_2) + (3b_1 - 5b_2) + (3c_1 - 5c_2) e_2,\]
\[\ker (I - A^{-1} M) = \langle e_1 \rangle = \begin{bmatrix} -1 \\ 1 \end{bmatrix},\]
\[-(I + MA) c = 2c_1 e_3,\]
\[\ker (A^{-1} + M) = \langle e_4 \rangle = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.\]

Since
\[(b_1 - b_2) e_1 + ((3a_1 - 5a_2) + (3b_1 - 5b_2) + (3c_1 - 5c_2)) e_2 + 2c_1 e_3 + (2a_1 + 2a_2) e_4\]
\[= a_1 (3e_2 + 2e_4) - a_2 (5e_2 - 2e_4) + b_1 (e_1 + 3e_2) - b_2 (e_1 + 5e_2) + c_1 (3e_2 + 2e_3) - 5c_2 e_2,\]
It follows that the Abelian group $H^2_\phi (Q_8; \mathbb{Z}^2)$ is isomorphic to
\[
\left\{ \begin{array}{c|ccc}
  e_1, e_2, e_3, e_4 & 3e_2 + 2e_4 = 0, & 5e_2 - 2e_4 = 0, & e_1 + 3e_2 = 0, \\
  e_1 + 5e_2 = 0, & 3e_2 + 2e_3 = 0, & 5e_2 = 0 \end{array} \right\}
\]
\[= \langle e_1', e_2', e_3', e_4' \mid e_1' = e_2' = 2e_1' = 2e_2' = 0 \rangle.\]
where \( \mathbf{e}_1' = \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2' = \mathbf{e}_2, \mathbf{e}_3' = \mathbf{e}_3, \mathbf{e}_4' = \mathbf{e}_4 \). We may choose
\[
\mathbf{k} + (\mathbf{n} + \mathbf{k}) = 0, \quad \mathbf{n} + \mathbf{k} = 0, \quad \mathbf{k}' - \mathbf{k} = 0 \quad \text{or} \quad \mathbf{e}_3, \quad \mathbf{m}' - \mathbf{n} + M(\mathbf{k}' - \mathbf{k}) = 0 \quad \text{or} \quad \mathbf{e}_4.
\]
Therefore,
\[
(\mathbf{m}', \mathbf{k}, \mathbf{k}', \mathbf{n}) = (0, 0, 0, 0), \quad (\mathbf{e}_3, 0, 0, 0), \quad (-M\mathbf{e}_3, 0, \mathbf{e}_3, 0), \quad (-M\mathbf{e}_3 + \mathbf{e}_4, 0, \mathbf{e}_3, 0).
\]
This yields the 4 inequivalent extensions \( E_8(0, \mathbf{m}', \mathbf{k}, \mathbf{k}', \mathbf{n}) \).

The following is an example that
\[
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\]

**Example 7.5** Let \( A = \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} \) and \( M = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \). Then \( \det M = -1, \text{tr } M = 0, MAM^{-1} = A^{-1} \). We observe that
\[
\ker(I - M) = (e_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix},
\]
\[
(I + M)b = b_2 e_1,
\]
\[
\ker(I + M) = (e_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]
\[
-(I - M)(a + b + c) = ((-2a_1 + a_2) + (-2b_1 + b_2) + (-2c_1 + c_2))e_2,
\]
\[
\ker(I - A^{-1}M) = (e_3) = \begin{bmatrix} -1 \\ 4 \end{bmatrix},
\]
\[
-(I + MA)c = -(2c_1 + c_2)e_3,
\]
\[
\ker(A^{-1} + M) = (e_4) = \begin{bmatrix} 3 \\ 2 \end{bmatrix},
\]
\[
(A - M)a = (4a_1 + a_2)e_4.
\]
It follows that the Abelian group \( H_2^Z(Q_8; Z^2) \) is isomorphic to
\[
\begin{bmatrix}
\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3', \mathbf{e}_4' \\
\mathbf{e}_1' + \mathbf{e}_2' = 0, \quad \mathbf{e}_2' + \mathbf{e}_3' = 0, \quad \mathbf{e}_2' - \mathbf{e}_3' = 0
\end{bmatrix}
\]
where \( \mathbf{e}_1' = \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2' = \mathbf{e}_2 + \mathbf{e}_4, \mathbf{e}_3' = \mathbf{e}_3, \mathbf{e}_4' = \mathbf{e}_3 + \mathbf{e}_4 \). We may choose
\[
\mathbf{k} + (\mathbf{n} + \mathbf{k}) = 0, \quad (\mathbf{n} + \mathbf{k}) + (\mathbf{m}' - \mathbf{n} + M(\mathbf{k}' - \mathbf{k})) = 0,
\]
\[
\mathbf{k}' - \mathbf{k} = 0 \quad \text{or} \quad \mathbf{e}_3, \quad (\mathbf{k}' - \mathbf{k}) + (\mathbf{m}' - \mathbf{n} + M(\mathbf{k}' - \mathbf{k})) = 0.
\]
Therefore,
\[
(\mathbf{m}', \mathbf{k}, \mathbf{k}', \mathbf{n}) = (0, 0, 0, 0) \quad \text{or} \quad (-3\mathbf{e}_3 - M\mathbf{e}_3, \mathbf{e}_3, 2\mathbf{e}_3, -2\mathbf{e}_3).
\]
This yields the 2 inequivalent extensions \( E_8(0, \mathbf{m}', \mathbf{k}, \mathbf{k}', \mathbf{n}) \).

### 7.6 Extensions by \( Q_9 \)

The extensions \( E_9 \) of \( Z^2 \) by \( Q_9 \) have presentations of the form
\[
E_9(k, k') = \begin{bmatrix}
a_1, a_2, \alpha, \beta \\
\begin{array}{c}
[a_1, a_2] = 1, \\
\alpha a_i \alpha^{-1} = \phi(\alpha)(a_i) \\
\beta a_i \beta^{-1} = \phi(\beta)(a_i) \\
\beta^2 = a_{i_1} a_{i_2} b_{i_1} b_{i_2} a_{i_3} \alpha^{-1}
\end{array}
\end{bmatrix}
\]
where
\[
\phi(\alpha) = - \begin{bmatrix}
\frac{\ell_{11} + 1}{\ell_{11} + \ell_{22} + 2} & \frac{\ell_{12}}{\ell_{21} + \ell_{22} + 2} \\
\frac{\ell_{12}}{\ell_{21} + \ell_{22} + 2} & \frac{\ell_{22} + 1}{\ell_{11} + \ell_{22} + 2}
\end{bmatrix} = N,
\]
\[
\phi(\beta) = M \in \text{GL}(2, \mathbb{Z}) \quad \text{where } \det M = -1, \text{tr} M = 0 \text{ and } MAM^{-1} = A^{-1}.
\]

Note that $MAM^{-1} = A^{-1}$ induces $MNM^{-1} = N^{-1}$, and the integer matrices $K = M - N$ and $L = M + N^{-1}$ satisfy the conditions of Lemma 7.2.

By Proposition 6.2, we have the following:

- The computational consistency conditions are
  \[(I - M)k = 0, \quad (I - N)k = (M - N)k'.\]
- The coboundary conditions are
  \[k = (I + M)c(\beta), \quad k' = (I - N^{-1})c(\beta) + (M + N^{-1})c(\alpha)\]
  for some 1-cochain $c : Q_9 \to \mathbb{Z}^2$.

Thus the computational consistency conditions become
\[(I - M)k = 0, \quad (M - N)(k' - k) = 0\]
with coboundary conditions
\[k = (I + M)c(\beta), \quad k' - k = (M + N^{-1})(c(\alpha) - c(\beta)).\]

Consequently
\[(k, k' - k) \in H_\delta^2(Q_9; \mathbb{Z}^2) = \frac{\ker(I - M)}{\text{im}(I + M)} \oplus \frac{\ker(N - M)}{\text{im}(N^{-1} + M)}.
\]

These information will give us all the inequivalent extensions $E_0(k, k')$. By Lemma 7.2, $H_\delta^2(Q_9; \mathbb{Z}^2)$ is finite. The extensions corresponding to the choice $k = 0$ are not torsion-free, and the rest extensions are torsion-free.

On the other hand, since $I + N$ and $I + N^{-1}$ are invertible, we have
\[(I - N)k = (M - N)k' \Rightarrow (I - A)k = (M - A)(I + N^{-1})k',
\]
\[k' = (I - N^{-1})c(\beta) + (M + N^{-1})c(\alpha)
\]
\[\Rightarrow (I + N^{-1})k' = (I - A^{-1})c(\beta) + (M + A^{-1})(I + N)c(\alpha).
\]

Thus the computational consistency conditions become
\[(I - M)k = 0, \quad (M - A)((I + N^{-1})k' - k) = 0\]
with coboundary conditions
\[k = (I + M)c(\beta), \quad (I + N^{-1})k' - k = (M + N^{-1})(c(\alpha) - c(\beta)).\]

Consequently in an equivalent way to the above, we have
\[(k, (I + N^{-1})k' - k) \in H_\delta^2(Q_9; \mathbb{Z}^2) = \frac{\ker(I - M)}{\text{im}(I + M)} \oplus \frac{\ker(A - M)}{\text{im}(A^{-1} + M)}.
\]
Notice that $Q_3 = \langle \alpha^2, \beta \rangle \subset Q_9$, and

$$\beta \alpha^2 \beta^{-1} = (\beta \alpha \beta^{-1})^2 = (a_1^{k_1} a_2^{k_2} \alpha^{-1})^2 = a_1^{k_1} a_2^{k_2} \phi(\alpha)^{-1}(a_1)^{k_1} \phi(\alpha)^{-1}(a_2)^{k_2} \alpha^{-2}.$$ 

Hence the extensions $E_0(k, k')$ of $\mathbb{Z}^2$ by $Q_9$ restrict to extensions $E_{3}(k, (I + N^{-1}) k')$ of $\mathbb{Z}^2$ by $Q_3$. This means the transfer (corestriction) homomorphism $i^* : H^2_{\phi}(Q_9; \mathbb{Z}^2) \rightarrow H^2_{\phi}(Q_3; \mathbb{Z}^2)$ sends $(k, k' - k)$ to $(k, (I + N^{-1}) k')$.

### 7.7 Extensions by $Q_{10}$

The extensions $E_{10}$ of $\mathbb{Z}^2$ by $Q_{10}$ have presentations of the form

$$E_{10}(k, k') = \left\{ a_1, a_2, \alpha \left| \begin{array}{c} [a_1, a_2] = 1, \\
i a_i t^{-1} = A(a_i) \\
\alpha a_i \alpha^{-1} = \phi(\alpha)(a_i) \\
\alpha^4 = a_1^{k_1} a_2^{k_2}, \ \alpha t \alpha^{-1} = a_1^{k_1} a_2^{k_2} t^{-1} \end{array} \right. \right\},$$

$$\phi(\alpha) = M \in \text{GL}(2, \mathbb{Z}) \quad \text{where} \quad \det M = 1, \ \text{tr} M = 0 \quad \text{and} \quad MAM^{-1} = A^{-1}.$$ 

By Proposition 6.2, we have the following:

- The computational consistency conditions are
  $$(I - M) k = 0, \quad (I - A) k = (M - A)(M^2 + I) k'.$$

- The coboundary conditions are
  $\begin{align*}
k &= f_k(\alpha^2, \alpha^2) = (I + M^2) c(\alpha^2), \\
0 &= f_k(\alpha, \alpha^2) = M c(\alpha^2) - c(\alpha^3) + c(\alpha), \\
0 &= f_k(\alpha^2, \alpha) = M^2 c(\alpha) - c(\alpha^3) + c(\alpha^2), \\
k' &= f_k(\alpha, t) = (M + A^{-1}) c(t) + (I - A^{-1}) c(\alpha)
  \end{align*}$$

for some 1-cocochain $c : Q_{10} \rightarrow \mathbb{Z}^2$.

Since the conditions on $M$ are forced to $M^2 = -I$, we have that $(I - M) k = 0 \Rightarrow (I - M^2) k = 0 \Rightarrow k = 0$. Moreover, the 1-cocochain $c : Q_{10} \rightarrow \mathbb{Z}^2$ must satisfy $c(\alpha^2) = (I + M) c(\alpha)$ and $c(\alpha^3) = M c(\alpha)$. Consequently

$$k = 0,$$

$$k' \in H^2_{\phi}(Q_{10}; \mathbb{Z}) = \frac{\mathbb{Z}^2}{\text{im} (M + A^{-1}) + \text{im} (I - A^{-1})}.$$ 

These information will give us all the inequivalent extensions $E_{10}(0, k')$. Note also that $H^2_{\phi}(Q_{10}; \mathbb{Z})$ is finite. All the extensions are not torsion-free.

Note that $Q_1 = \{ t, \alpha^2 \} \subset Q_{10}$, and $\alpha^2 t \alpha^{-2} = (a_1^{k_1} a_2^{k_2} t^{-1}) \alpha^{-1} = \phi(\alpha)(a_1)^{k_1} \phi(\alpha)(a_2)^{k_2} t a_1^{-k_1} a_2^{-k_2} = \phi(\alpha)(a_1)^{k_1} \phi(\alpha)(a_2)^{k_2} A(a_1)^{-k_1} A(a_2)^{-k_2} t$. Hence the extensions $E_{10}(0, k')$ of $\mathbb{Z}^2$ by $Q_{10}$ restrict to extensions $E_{1}(0, (M - A) k')$ of $\mathbb{Z}^2$ by $Q_{1}$. This means the transfer (corestriction) homomorphism $i^* : H^2_{\phi}(Q_{10}; \mathbb{Z}^2) \rightarrow H^2_{\phi}(Q_{1}; \mathbb{Z}^2)$ sends $k'$ to $(M - A) k'$. 

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7.8 Extensions by \( Q_{11} \)

Note that \( Q_{11} \) has another presentation \( Q_{11} = \{ \alpha, \beta \mid \alpha^2 = 1, \alpha \beta^2 \alpha^{-1} = \beta^{-2}, \alpha \beta^{-1} = \beta \alpha^{-1} \} \). Every element of \( Q_{11} \) can be written as \( \beta^a \alpha^b \). The extensions \( E_{11} \) of \( \mathbb{Z}^2 \) by \( Q_{11} \) have presentations of the form

\[
E_{11}(k, p, q) = \begin{pmatrix}
[a_1, a_2] = 1, \\
\alpha a_i a^{-1} = \phi(\alpha)(a_i) \\
\beta a_i \beta^{-1} = \phi(\beta)(a_i) \\
\alpha^3 = a_1^i a_2^2, \ \alpha \beta^2 \alpha^{-1} = a_1^i a_2^2 \beta^{-2}, \\
\alpha \beta^{-1} = a_1^i a_2^2 \beta \alpha^{-1}
\end{pmatrix}
\]

\( \phi(\alpha) = M \in \text{GL}(2, \mathbb{Z}) \) where \( \det M = 1, \text{tr} M = 0 \) and \( MAM^{-1} = A^{-1} \),

\( \phi(\beta) = \begin{pmatrix}
\ell_{11} - 1 \\
\sqrt{\ell_{11} + \ell_{22} - 2} \\
\ell_{21} \\
\sqrt{\ell_{11} + \ell_{22} - 2}
\end{pmatrix} N \in \text{GL}(2, \mathbb{Z}). \)

Note that \( M^2 = -I \) and \( MNM^{-1} = -N^{-1} \). By Proposition 6.2, we have the following:

- The computational consistency conditions are

\[
(I - M)k = 0 \Rightarrow (I - M^2)k = (I + M)0 = 0 \Rightarrow as M^2 = -I, k = 0,
(I - A)k = 0, (M + N^{-1})q = 0.
\]

- The coboundary conditions are

\[
k = f_k(\alpha^2, \beta^2) = (I + M^2)c(\alpha^2) = 0 as M^2 = -I,
q = f_k(\alpha, \beta^{-1}) = Mc(\beta^{-1}) - c(\beta \alpha^{-1}) + c(\alpha),
0 = f_k(\alpha^{-1}, \alpha) = M^{-1}c(\alpha) + c(\alpha) \rightarrow c(\alpha^{-1}) = -M^{-1}c(\alpha) = Mc(\alpha),
0 = f_k(\beta^{-1}, \beta) = N^{-1}c(\beta) + c(\beta^{-1}) \rightarrow c(\beta^{-1}) = -N^{-1}c(\beta),
0 = f_k(\beta, \alpha^{-1}) = Nc(\alpha^{-1}) - c(\beta \alpha^{-1}) + c(\beta) \rightarrow c(\beta \alpha^{-1}) = Nc(\alpha^{-1}) + c(\beta),
\]

\[
\therefore q = Mc(\beta^{-1}) - c(\beta \alpha^{-1}) + c(\alpha),
\]

\[
\therefore q = -M N^{-1}c(\beta) - N Mc(\alpha) - c(\beta) + c(\alpha)
\]

\[
\therefore q = (N M - I)(c(\beta) - c(\alpha)) = (N + M)M(c(\beta) - c(\alpha)),
\]

\[
p = f_k(\alpha, \beta^2) = Mc(\beta^2) - c(\beta^{-2} \alpha) + c(\alpha),
\]

\[
0 = f_k(\beta, \beta) = Nc(\beta) - c(\beta^2) + c(\beta) \rightarrow c(\beta^2) = (I + N)c(\beta),
0 = f_k(\beta^{-1}, \beta^{-1}) = N^{-1}c(\beta^{-1}) - c(\beta^{-2}) + c(\beta^{-1}) \rightarrow c(\beta^{-2}) =
\]

\[
(I + N^{-1})c(\beta^{-1}).
\]

\[
0 = f_k(\beta^{-2}, \alpha) = N^{-2}c(\alpha) - c(\beta^{-2} \alpha) + c(\beta^{-2}) \rightarrow c(\beta^{-2} \alpha) = A^{-1}c(\alpha) + c(\beta^{-2}).
\]

\[
\therefore p = Mc(\beta^2) - c(\beta^{-2} \alpha) + c(\alpha)
\]

\[
= M(I + N)c(\beta) - A^{-1}c(\alpha) + (I + N^{-1})N^{-1}c(\beta) + c(\alpha)
\]

\[
= (I - A^{-1})c(\alpha) + (M + A^{-1})(I + N)c(\beta)
\]

for some 1-cochain \( c : Q_{11} \rightarrow \mathbb{Z}^2 \).
Therefore the conditions are reduced to the following:

- The computational consistency conditions are

  \[ k = 0, \quad (M + N^{-1})q = 0. \]

- The coboundary conditions are

  \[ k = 0, \]

  \[ q = (M + N)M(b - a) = (M + N)x, \]

  \[ p = (I - A^{-1})a + (M + A^{-1})(I + N)b \]

  \[ = (I - A^{-1})a + (M + A^{-1})(I + N)(a - Mx) \]

  \[ = ((I - A^{-1}) + (M + A^{-1})(I + N))a - (M + A^{-1})(I + N)Mx \]

  where \( x = M(b - a) \) and so \( b = a - Mx \).

Recalling that \( M^2 = -I \) and \( MNM^{-1} = -N^{-1} \), we see that \( (M + N^{-1})(M + N) = 0 \). We can also check easily that \( K = M + N^{-1} \) and \( L = M + N \) satisfy the conditions of Lemma 7.2. Since for any \( x \) we can regard \( \mathbb{Z}^2 \) as \( \mathbb{Z}^2 = -(M + A^{-1})(I + N)Mx + \mathbb{Z}^2 \), we have

\[ (p, q) \in H^2_\phi(Q_{11}; \mathbb{Z}^2) \cong \frac{\mathbb{Z}^2}{\text{im}(I - A^{-1}) + (M + A^{-1})(I + N)} \oplus \ker(M + N^{-1}) \]

and all the inequivalent extensions are \( E_{11}(p, q) \) where \( (p, q) \in H^2_\phi(Q_{11}; \mathbb{Z}^2) \). We can show that the determinant of \( (I - A^{-1}) + (M + A^{-1})(I + N) \) is zero, and hence \( H^2_\phi(Q_{11}; \mathbb{Z}^2) \) is infinite. These extensions are not torsion-free.

Note that \( Q_{11} \) has a subgroup generated by \( \beta^2 \) and \( \alpha \), which is isomorphic to \( Q_{10} \). The extensions \( E_{11}(p, q) \) of \( \mathbb{Z}^2 \) by \( Q_{11} \) restrict to extensions \( E_{10}(0, p) \) of \( \mathbb{Z}^2 \) by \( Q_{10} \). This means that the inclusion \( i : Q_{10} \to Q_{11} \) induces the homomorphism \( i^* : H^2_\phi(Q_{11}; \mathbb{Z}^2) \to H^2_\phi(Q_{10}; \mathbb{Z}^2) \) which is a surjection sending \( (p, q) \) to \( p \in \mathbb{Z}^2/(\text{im}(I - A^{-1}) + \text{im}(M + A^{-1})) \).

**Example 7.6**

Consider

\[ A = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}, \quad N = -\begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix}. \]

Then \( \det M = 1, \text{tr}M = 0, MAM^{-1} = A^{-1} \) and \( N^2 = A \). Since \( \det(A^{-1} - I) = -1 \), it follows that

\[ H^2_\phi(Q_{10}; \mathbb{Z}) = \mathbb{Z}^2/(\text{im}(M + A^{-1}) + \text{im}(I - A^{-1})) = \{0\}. \]

Hence there is a unique extension \( E_{10} \) of \( \mathbb{Z}^2 \) by \( Q_{10} \).

On the other hand, a simple computation shows that \( \ker(M + N^{-1}) = \text{im}(M + N) \) and \( (I - A^{-1}) + (M + A^{-1})(I + N) \) has determinant 0. In fact, \( \mathbb{Z}^2/\text{im}(I - A^{-1}) + (M + A^{-1})(I + N) \cong \mathbb{Z} \). Thus there are infinitely many inequivalent extensions \( E_{11} \) of \( \mathbb{Z}^2 \) by \( Q_{11} \), all of which restrict to the unique extension \( E_{10} \).

**8 Conclusions**

Finally, we move to Step 3. We first check whether the abstract kernels \( \rho : \Phi \to \text{Out}(\Gamma_4) \) of the extensions \( E \) are injective. This follows by observing how the abstract kernel is defined and then by using Lemma 3.8. In fact, for

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any \( g \in E \) if \( g \mapsto \alpha \in \Phi \) and \( g \mapsto q \in Q \) then we have the commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \mathbb{Z}^2 \\
\downarrow{\phi(q)} & & \downarrow{\text{conj}(q)} \\
0 & \longrightarrow & \mathbb{Z}^2 \\
\end{array}
\]

By definition,

\[
\rho(\alpha) = \text{conj}(g) = \begin{bmatrix} \phi(q) & \psi(\alpha) \\ 0 & \psi(\alpha) \end{bmatrix}
\]

\[
\equiv \begin{bmatrix} \phi(\alpha) & \psi(\alpha) \\ 0 & \psi(\alpha) \end{bmatrix} \mod \text{Inn}(\Gamma_A).
\]

Now by Lemma 5.9 we know for each \( E \) (or \( Q \)) the induced actions \( \phi \) and \( \psi \), and then from Lemma 3.8 we see easily that \( \rho \) is injective.

Let us compute now the centralizer \( C_{E_i}(\Gamma_A) \) of \( \Gamma_A \) in \( E_i \).

**Theorem 8.1** We have \( C_{E_i}(\Gamma_A) = 0 \) for all \( i \).

**Proof.** If \( i = 0 \) then since \( E_0 = \Gamma_1, C_{E_i}(\Gamma_A) = Z(\Gamma_A) = 0 \) by Lemma 3.6.

Let \( g \in C_{E_i}(\Gamma_A) \). Then \( g \gamma g^{-1} = \gamma \) for all \( \gamma \in \Gamma_A \). If \( g \in \Gamma_A \), then \( g \in Z(\Gamma_A) \) and by Lemma 3.6, \( g = 1 \). So, we have to assume \( g \not\in \Gamma_A \) and show it is impossible.

Suppose \( i = 5 \) and \( g \in C_{E_i}(\Gamma_A) - \Gamma_A \). Then \( g \) can be expressed as \( g = a_i^n a^2_i a^k \beta^l \). Since \( g \not\in \Gamma_A \), whenever \( k \) is even, \( \ell \) must be odd. This element commutes with \( a_i \). The equalities \( (a_i^n a^2_i a^k \beta^l a_i a_i^n a^2_i a^k \beta^l)^{-1} = a_i (i = 1, 2) \) induce \((-1)^k N^\ell = I \) where \( N \) is a square root of \( \alpha \). If \( \ell = 0 \) then \( k \) is even and so \( \ell \neq 0 \). Thus \( N^\ell = (-1)^k I \) with \( \ell \neq 0 \). But this is impossible because \( N \) has irrational eigenvalues.

Suppose \( i = 8 \) and \( g \in C_{E_i}(\Gamma_A) - \Gamma_A \). Then \( g \) can be expressed as \( g = a_i^n a^2_i t^k a^q \beta^r \) where \( (p, q) = (1, 0) \) or \((0, 1) \) or \((1, 1) \). This element commutes with \( a_i \). The equalities \( (a_i^n a^2_i t^k a^q \beta^r a_i a_i^n a^2_i t^k a^q \beta^r)^{-1} = a_i (i = 1, 2) \) induce \( A^k (-1)^r M^q = I \) where \( M^2 = I \). We can see \( k \neq 0 \). Hence we obtain a contradiction because \( A \) has irrational eigenvalues.

The arguments for the rest cases are identical.

We close this section by stating the following theorems which summarize our results.

**Theorem 8.2** There are 9 kinds of SC-groups.

(1) \( E_0 = \left\{ a_1, a_2, t \mid [a_1, a_2] = 1, ta_t^{-1} = A(a_1) \right\} = \Gamma_A \).

\( \Phi = \{1\} \).

(2) \( E_1(\mathbf{k}) = \left\{ a_1, a_2, t, \beta \mid [a_1, a_2] = 1, ta_t^{-1} = A(a_1), \beta a_t \beta^{-1} = a^{-1}_i \right\} \)

where \( \mathbf{k} \in H_2^2(Q_1; \mathbb{Z}^2) \cong \mathbb{Z}^2/\left(2(\mathbb{Z}^2) + \text{im}(I - A)\right) \).

\( \Phi = \left[ \begin{array}{ccc} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right] \subset \text{Aut}(\text{Sol}) \).

(3) \( E_2^\pm = \left\{ a_1, a_2, \beta \mid [a_1, a_2] = 1, \beta a_t \beta^{-1} = N_{\pm}(a_1) \right\} \)

where \( A \) has a square root \( N \).

\( N_+ = \begin{bmatrix} \ell_{11} + 1 \\ \sqrt{\ell_{11} + \ell_{22} + 2} \\ \sqrt{\ell_{21} + \ell_{22} + 2} \end{bmatrix}, \quad N_- = \begin{bmatrix} \ell_{11} - 1 \\ \sqrt{\ell_{11} + \ell_{22} - 2} \\ \sqrt{\ell_{21} + \ell_{22} - 2} \end{bmatrix}. \)
\[ \Phi = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \subset \text{Aut(Sol)} \quad \text{and} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \subset \text{Aut(Sol)}, \text{respectively.} \]

(4) \( E_3(k, k') = \left\{ a_1, a_2, t, \alpha, \beta \mid [a_1, a_2] = 1, \, t a_1 t^{-1} = A(a_1), \, \beta a_1 \alpha^{-1} = M(a_1), \right\} \)

where \( M \) is traceless with determinant \(-1\) and \( M A M^{-1} = A^{-1} \), and

\( (k, k' - k) \in H_\Phi^2(Q_5; \mathbb{Z}^2) \cong \ker(I - M)/\ker(A - M)/\ker(A^{-1} + M). \)

\[ \Phi = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \subset \text{Aut(Sol)}. \]

(5) \( E_2(k) = \left\{ a_1, a_2, \alpha, \beta \mid [a_1, a_2] = 1, \, \alpha a_1 \alpha^{-1} = a_1^{-1}, \, \alpha^2 = 1, \quad \right\} \)

where \( A \) has a square root \( N \)

\[ N = -\begin{bmatrix} \ell_{11} - 1 & \ell_{12} \\ \sqrt{\ell_{11} + \ell_{22} - 2} & \sqrt{\ell_{11} + \ell_{22} - 2} \\ -\ell_{21} & \ell_{22} - 1 \\ \sqrt{\ell_{11} + \ell_{22} - 2} & \sqrt{\ell_{11} + \ell_{22} - 2} \end{bmatrix} \]

and

\( k \in H_\Phi^2(Q_5; \mathbb{Z}^2) \cong \mathbb{Z}^2/(2(\mathbb{Z}^2) + \ker(I - N)). \)

\[ \Phi = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \subset \text{Aut(Sol)}. \]

(6) \( E_3(m, k, k', n) = \left\{ a_1, a_2, t, \alpha, \beta \mid [a_1, a_2] = 1, \, t a_1 t^{-1} = A(a_1), \right\} \)

where \( M \) is traceless with determinant \(-1\) and \( M A M^{-1} = A^{-1} \), and

\( (k, n + k' - k, m - n + M(k' - k)) \in H_\Phi^2(Q_5; \mathbb{Z}^2) \)

\[ \cong \text{ker}(I - M)/\text{im}(I + M) + \text{ker}(I - A^{-1} M)/\eta \times \text{im}(I + MA) + \text{ker}(A^{-1} + M)/\text{im}(A - M). \]

Here \( \eta = 1 \) or \( 2 \), and \( \eta = 2 \) if and only if \( A \) and \( M \) can be conjugated simultaneously to

\[ \begin{bmatrix} \ell_{11}' & \ell_{12}' \\ \ell_{21}' & \ell_{22}' \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \]

so that both \( \frac{\ell_{11}'}{\text{gcd}(\ell_{11}', \ell_{21}')} \) and \( \frac{\ell_{22}'}{\text{gcd}(\ell_{22}', \ell_{21}')} \) are even.

\[ \Phi = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \subset \text{Aut(Sol)}. \]
\[
E^i(k, k') = \begin{cases}
(a_1, a_2, \alpha, \beta) & [a_1, a_2] = 1, \ a\alpha a^{-1} = M(a_i), \\
[\alpha^x = 1, \ a\alpha^{-1} = a_1^x a_2^x] & \alpha^x = 1, \ a\alpha^{-1} = a_1^x a_2^x \end{cases}
\]

where \( A \) has a square root \( N \)

\[
N = -\begin{bmatrix}
\ell_{11} + 1 & \ell_{12} \\
\sqrt{\ell_{11} + \ell_{22} + 2} & \sqrt{\ell_{11} + \ell_{22} + 2}
\end{bmatrix},
\]

and \( M \) is traceless with determinant \(-1\) and \( \text{MAM}^{-1} = A^{-1} \), and

\[
(k, k' - k) \in H^2_p(Q; \mathbb{Z}^2) \cong \ker(I - M)/\text{im}(I + M) \oplus \ker(N - M)/\text{im}(N^{-1} + M).
\]

\[
\Phi = \begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix} \subset \text{Aut}(\text{Sol}).
\]

\[
E^{10}(k) = \begin{cases}
(a_1, a_2, \alpha, \beta) & [a_1, a_2] = 1, \ a\alpha a^{-1} = M(a_i), \\
[a_1^\alpha = 1, \ a\alpha^{-1} = a_1^x a_2^x] & \alpha^x = 1, \ a\alpha^{-1} = a_1^x a_2^x \end{cases}
\]

where \( M \) is traceless with determinant \(1\) and \( \text{MAM}^{-1} = A^{-1} \) and

\[
k \in H^2_p(Q; \mathbb{Z}^2) \cong \mathbb{Z}^2/(\text{im}(M + A^{-1}) + \text{im}(I - A^{-1})).
\]

\[
\Phi = \begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} \subset \text{Aut}(\text{Sol}).
\]

\[
E^{11}(k, m) = \begin{cases}
(a_1, a_2, \alpha, \beta) & [a_1, a_2] = 1, \ a\alpha a^{-1} = M(a_i), \\
[a_1^\alpha = 1, \ a\alpha^{-1} = a_1^x a_2^x \beta^{-2}, \ a\beta^{-1} = a_1^m a_2^m \beta^{-1}] & \alpha^x = 1, \ a\alpha^{-1} = a_1^x a_2^x \end{cases}
\]

where \( A \) has a square root \( N \)

\[
N = -\begin{bmatrix}
\ell_{11} - 1 & \ell_{12} \\
\sqrt{\ell_{11} + \ell_{22} - 2} & \sqrt{\ell_{11} + \ell_{22} - 2}
\end{bmatrix},
\]

and \( M \) is traceless with determinant \(1\) and \( \text{MAM}^{-1} = A^{-1} \), and

\[
(k, m) \in H^2_p(Q; \mathbb{Z}^2) \cong \mathbb{Z}^2/(\text{im}(I - A^{-1}) + (M + A^{-1})(I + N)) \oplus \ker\left(\text{im}(M + N^{-1})\right).
\]

\[
\Phi = D(4) = \begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} \subset \text{Aut}(\text{Sol}).
\]

Remark that the extensions \( E_4, E_5, E_6, E_{10} \) and \( E_{11} \) are not torsion-free, and the extensions \( E_9 \) and \( E_2 \) are always torsion-free. For the remaining extensions \( E_3 \) and \( E_9 \), if \( k = 0 \) then they are not torsion-free.

Regarding the extensions \( E_5(k, k') \) and \( E_9(k, k') \), we may assume by Remark 5.7 that

\[
M = \begin{bmatrix}
-1 & \ell_{11} - \ell_{22} \\
\ell_{21} & 1
\end{bmatrix}
\]
where $\ell_{11} - \ell_{12} = 0$ or 1. If $\ell_{11} - \ell_{12} = 0$, using Lemma 7.2 we can show easily that $\ker(I - M) / \text{im}(I + M) \cong \mathbb{Z}_2$ with a generator $k = [0, 1]$. Note that $Mk = k$. On the other hand, if $\ell_{11} - \ell_{12} = 1$, then we can show that $\ker(I - M) / \text{im}(I + M) = \{0\}$ and $k = 0$. In $E_3$, if $k = k'$ then $\beta^2 = a^k, \beta t^{-1} = a^k t^{-1}$ induce that $\beta t^{-1} = \beta^2 t^{-1}$ and so $t^{-1} = \beta t^{-1}$. Thus $t \beta^{-1}$ is an element of order 2. (If $k = (A^{-1} + M)p$ then $a^{-p} t \beta^{-1}$ has order 2. For, $(a^{-p} t \beta^{-1})^2 = (a^{-p} t \beta^{-1}) (a^{-p} t \beta^{-1}) = a^{-p} t \beta^{-1} a^{-p} t \beta^{-1} = a^{-p} t a^{-k} a^{-Mp+k} t^{-1} = a^{-p} t a^{-k} - A M p + A k = A (k' - k - (A^{-1} + M)p) = 1$.) Similarly, in $E_0$ if $k = k'$ then $s \beta^{-1}$ is a torsion element of order 2. (If $k' - k \in \text{im}(N^{-1} + M)$ then $E_0$ is not torsion-free.)

Therefore, to have torsion-free extensions $E_3$ or $E_9$, we must have $\ell_{11} - \ell_{12} = 0$, or equivalently $\ell_{11} = \ell_{22}$, and $k = [0, 1]'$, and $k' - k \neq 0$ in $\ker(A - M) / \text{im}(A^{-1} + M)$ or $\ker(N - M) / \text{im}(N^{-1} + M)$ respectively. In $E_9$, since $\sqrt{2}(\ell_{11} + 1) = 2s$ for some $s$, we have that $\ell_{11} = 2s^2 - 1$ with $s > 1$ and $\ell_{12} = 2sp$ and $\ell_{21} = 2sq$ with $pq = s^2 - 1$. Hence

$$A = \begin{bmatrix} 2s^2 - 1 & 2sp \\ 2sq & 2s - 1 \end{bmatrix}, \quad N = \begin{bmatrix} s & p \\ q & s \end{bmatrix}. $$

Notice that $A$ of $E_3$ and $N$ of $E_9$ have the same form. By Lemma 7.2, $\ker(A - M) / \text{im}(A^{-1} + M) \cong \mathbb{Z}_2$ and $\ker(N - M) / \text{im}(N^{-1} + M) \cong \mathbb{Z}_2$ where $g = \gcd(\ell_{11} + 1, \ell_{12}, \ell_{21})$ and $h = \gcd(s \pm 1, p, q)$. Hence $g$ and $h$ are 1 or 2. To have $k' - k \neq 0$, $g$ and $h$ must be 2, or equivalently, $\ell_{11}$ is odd, $\ell_{12}, \ell_{21}$ are even, $s$ is odd and $p, q$ are even.

Now we claim that the converse also holds. For notational simplicity, let us consider only $E_3$. Namely, if $\ell_{11} = \ell_{22}$ is odd and $\ell_{12}, \ell_{21}$ are even, $k = [0, 1]'$, and $k' - k \in \ker(A - M)$ but $k' - k \notin \text{im}(A^{-1} + M)$, then $E_3$ is torsion-free. Since $E_3$ is an extension of $\Gamma_A$ by $\mathbb{Z}_2$, for any element, its square is in $\Gamma_A$. Hence a nontrivial torsion element must be of order 2. Every element of $E_3$ is of the form $a^p t^q \beta^r$ where $r = 0$ or $-1$. Assume it is a torsion element of order 2. Since $\Gamma_A$ is torsion-free, any torsion element of the form $a^p t^q \beta^r$ is the identity element. Consider the torsion element $a^p t^q \beta^r$. Then

$$1 = (a^p t^q \beta^r)^2 = (a^p t^q \beta^r)(a^p t^q \beta^{-r}) = a^{p+ApM} t^q \beta^{-r} t^q \beta^{-r}. $$

If $q = 0$ then the above identity becomes $1 = a^{(I + M)p - k}$ or $(I + M)p = k$. The $(1, 2)$-component of the left-hand side is even, but that of the right-hand side is 1. Hence every element of the form $a^p \beta^r$ is not a torsion element. For $q \neq 0$, from the observation that

$$t^{q-1} \beta^{-r} = \begin{cases} a^{M(I + A^{-1} + \ldots + A^{-(-q-1)})} k' \beta^{-1}, & \text{when } q > 0, \\ a^{-M + A^2 + \ldots + A^{-q}} k' \beta^{-1}, & \text{when } q < 0, \end{cases} $$

we have

$$1 = \begin{cases} a^{p+ApM} p a^{M(I + A^{-1} + \ldots + A^{-(-q-1)})} k' \beta^{-1}, & \text{when } q > 0, \\ a^{p+ApM} p a^{-M + A^2 + \ldots + A^{-q}} k' \beta^{-1}, & \text{when } q < 0. \end{cases} $$

Thus

$$(I + A^q) p = \begin{cases} k - M(I + A^{-1} + \ldots + A^{-(-q-1)}) k', & \text{when } q > 0, \\ k + M(A + A^2 + \ldots + A^{-q}) k', & \text{when } q < 0. \end{cases} $$

Since $Mk = k$ and $M^2 = I$, we have

$$(M + A^{-q}) p = \begin{cases} k - (I + A^{-1} + \ldots + A^{-(-q-1)}) k', & \text{when } q > 0, \\ k + (A + A^2 + \ldots + A^{-q}) k', & \text{when } q < 0. \end{cases} $$

All the entries of the left-hand side are even and if $q$ is even then the $(1, 2)$-component of the right-hand side is odd. Therefore, we may assume $q = 2p + 1$ is odd. If $p = 0$, then $(M + A^{-q}) p = k - k' \Rightarrow k' - k = (A^{-1} + M)(-p) \in \text{im}(A^{-1} + M)$, a contradiction. Now consider $p > 0$. Then

$$(M + A^{-q}) p = A^{-p}(I + A^{-1} + \ldots + A^{-2p}) k'$$

$$\Rightarrow (A^{-1} + M) A^{-p} p = A^p (k - k') - A^p (A^{-1} + A^{-2} + \ldots + A^{-2p}) k'. $$
Since $k' - k \in \ker(A - M)$ and $Mk = k$, we have $A^p(k - k') = A^{p-1}k - A^{p-1}MK' = A^{p-1}k - MA^{-(p-1)}k'$.

Inductively, we will have
\[
(A^{-1} + M)A^{-p}p = (k - k') - (A^{-1} + M)(I + A^{-1} + \cdots + A^{-(p-1)})k'.
\]

This shows that $k - k' \in \im(A^{-1} + M)$, a contradiction. Next consider $p < 0$. Then
\[
(M + A^{-q})p = A^{-p}(A^{-1} + M)A^{-p}p = k + (A + A^2 + \cdots + A^{-(2p+1)})k'
\]
\[
= (A^{p+1} + \cdots + A^{-1} + I + A + \cdots + A^{-(p+1)})(k' - k)
\]
\[
+ (A^p + A^{p+1} + \cdots + A^{-(p+1)})k
\]
\[
= (A^{p+1} + \cdots + A^{-1} + I + A + \cdots + A^{-(p+1)})(k' - k)
\]
\[
+ A^{-1}(A^{p+1} + A^{p+3} + \cdots + I)k + (I + A + \cdots + A^{-(p+1)})Mk
\]
\[
= (A^{p+1} + \cdots + A^{-1} + I + A + \cdots + A^{-(p+1)})(k' - k)
\]
\[
+ A^{-1}(A^{p+1} + A^{p+3} + \cdots + I)k + M(I + A^{-1} + \cdots + A^{p+1})k
\]
\[
= (A^{p+1} + \cdots + A^{-1} + I + A + \cdots + A^{-(p+1)})(k' - k)
\]
\[
+ (A^{-1} + M)(A^{p+1} + A^{p+2} + \cdots + I)k.
\]

Since $A^{-r} + A^r = \text{even integer} \times I$, we have odd integer $\times (k' - k) \in \im(A^{-1} + M)$. But this is impossible because $\im(A^{-1} + M)$ is generated by $2(k' - k)$.

With the above observations, we have:

**Corollary 8.3** There are 4 kinds of SB-groups.

1. $E_0 = \{a_1, a_2, t | [a_1, a_2] = 1, ta_t^{-1} = A(a_i)\} = \Gamma_A$.
2. $E_2^\pm = \{a_1, a_2, s | [a_1, a_2] = 1, sa_s^{-1} = N(a_i)\}$

where
\[
N_+ = -\begin{bmatrix}
\ell_{11} + 1 & \ell_{12} \\
\ell_{21} & \ell_{22} + 2
\end{bmatrix}
\quad \text{and} \quad
N_- = -\begin{bmatrix}
\ell_{11} - 1 & \ell_{12} \\
\ell_{21} & \ell_{22} - 2
\end{bmatrix}
\]
\[
\Phi = \left\begin{array}{ccc}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{array}\right\}
\quad \text{and} \quad
\left\begin{array}{ccc}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right\}
\]
respectively.

3. $E_3 = \{a_1, a_2, t, \alpha | [a_1, a_2] = 1, ta_t^{-1} = A(a_i), \alpha a_i \alpha^{-1} = M(a_i)\}$

where
\[
A = \begin{bmatrix}
\ell_{11} & \ell_{12} \\
\ell_{21} & \ell_{22}
\end{bmatrix}
\quad (\ell_{11} \text{ odd}; \ell_{12}, \ell_{21} \text{ even}),
\]
\[
M = \begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}
\]
\[
k = [0, 1]^	op, \quad k' with k' - k \neq 0 in \ker(A - M)/\im(A^{-1} + M) \cong \mathbb{Z}_2.
\]
\[
\Phi = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]

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\[ E_0 = \left\{ a_1, a_2, s, \alpha \middle| a_1a_2 = 1, \ sa_1a_2^{-1} = N(a_1), \ aa_1a_2^{-1} = M(a_1), \right\} \]

where

\[ A = \begin{bmatrix} 2s^2 - 1 & 2sp \\ 2sq & 2s^2 - 1 \end{bmatrix}, \ N = -\begin{bmatrix} s & p \\ q & s \end{bmatrix}, \ M = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \]

\((s > 1, s^2 - pq = 1, s \ odd, \ p, q \ even)\)

\[ k = [0, 1]^t, \ k' - k \neq 0 \text{ in } \ker(N - M)/\text{im}(N^{-1} + M) \cong \mathbb{Z}_2. \]

\[ \Phi = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \]

9 Comparison with known facts

The closed 3-manifolds with Sol-geometry have been studied by many authors, for example [21], [11], [19]. However, they are given in a slightly different form compared to ours. In this section, we will compare our exposition with some of known expositions.

Let \( f : N \to N \) be a diffeomorphism of a manifold \( N \). The action of \( \mathbb{Z} \), defined on \( N \times \mathbb{R} \) by

\[ (k, (x, t)) \mapsto (f^k(x), t + k) \]

yields the quotient manifold \( N_f \mathbb{R} \). This manifold is called the suspension of \( N \) defined by \( f \). A linear map \( \phi \in \text{GL}_2(\mathbb{Z}) \) is called hyperbolic if it has two distinct real eigenvalues different from \( \pm 1 \).

By [18, Lemma 2.3], the suspension of the torus by a hyperbolic linear map \( \phi \in \text{GL}(2, \mathbb{Z}) \) is diffeomorphic to the quotient \( \Lambda/\text{Sol} \) where \( \Lambda \subset \text{Isom(\text{Sol})} \) is an extension of a lattice \( \Lambda_0 \) of \( \mathbb{R}^2 \subset \text{Sol} \) by a subgroup \( \langle (\lambda, D) \rangle \) of \( \text{Aut}(\mathbb{R}^2) \) (see diagram (3.1)) where \( \lambda \in \mathbb{R}^* \) and

\[ D = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \text{Aut(\text{Sol})} \]

with \( \epsilon_1 = \pm 1 \). Note that if \( \epsilon_1 = \epsilon_2 = 1 \), then \( \Lambda \) is a lattice \( E_0 = \Gamma_\Lambda \) of \( \text{Sol} \) and if \( \epsilon_1 = -\epsilon_2 = \pm 1 \), then the corresponding \( \Lambda \)'s are isomorphic to \( E_1 \), and if \( \epsilon_1 = \epsilon_2 = -1 \), then \( \Lambda \) is isomorphic to \( E_2^\perp \). Consequently, the suspension of the torus by a hyperbolic linear map \( \phi \in \text{GL}(2, \mathbb{Z}) \) is diffeomorphic to the quotient \( E_0/\text{Sol} \) or \( E_2^\perp/\text{Sol} \). Note also that the quotient \( \Lambda/\text{Sol} \) has a bundle structure \( \Lambda/\text{Sol} \to ((\Lambda_0/\mathbb{Z})/\mathbb{R} \cong S^1. \)

By [18, Definition 2.2], a sapphire space is the quotient of a bundle \( \Lambda/\text{Sol} \to S^1 \) by an involutive isometry acting without fixed point and inducing a reflection on the base \( S^1 \). Note that \( E_0 \subset E_3 \) and \( E_2^\perp \subset E_3 \) with quotient isomorphic to the cyclic group \( \mathbb{Z}_2 \) of order 2 generated by

\[ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \in \text{Aut(\text{Sol})}. \]

Hence such sapphire spaces are exactly the quotients \( E_3/\text{Sol} \) and \( E_0/\text{Sol} \).

Consequently, we can recapture the following result [18, Theorem 2.1]: A closed 3-manifold \( M \) admits Sol-geometry if and only if \( M \) is the suspension of a diffeomorphism of the torus \( \mathbb{R}^2/\mathbb{Z}^2 \) defined by a hyperbolic linear map or a sapphire space.

In [11], the orientable closed 3-manifolds with Sol-geometry are studied. A torus bundle is \( M_\phi = T \times I/(x, 1) \sim (\phi(x), 0) \) where \( \phi \in \text{GL}(2, \mathbb{Z}) \) is a diffeomorphism of the torus \( T \). Let \( K \) be the Klein bottle and
$N = K \tilde{\times} I$ be the orientable twisted $I$-bundle over $K$. A torus semi-bundle $N_\phi = N \cup_\phi N$ is obtained by gluing two copies along their torus boundary $\partial N$ via a diffeomorphism $\phi$.

The torus semi-bundles always have a torus bundle as a 2-sheeted covering space. The torus bundles and torus semi-bundles which are not Seifert-fibered provide examples of manifolds with Sol-geometry. It is known (see for example [20, Propositions 1.3 and 1.5]) that an orientable $M_\phi$ admits Sol-geometry if and only if $\phi$ is conjugate to $B \in \text{SL}(2, \mathbb{Z})$ with $|\text{tr}B| > 2$, and $N_\phi$ admits Sol-geometry if and only if $\phi = B \in \text{SL}(2, \mathbb{Z})$ via a choice of coordinates in $\partial N$ where all the entries of $B$ are nonzero.

We remark that the torus bundles $M_\phi$ are the suspensions of the torus $T$ defined by $\phi$. On the other hand, the fundamental group $\pi$ of $M_\phi$ is isomorphic to $\mathbb{Z}^2 \times \mathbb{Z}$. If $\text{det} \phi = 1$ and $\text{tr} \phi > 2$, then $\pi \cong S_0$; if $\text{det} \phi = 1$ and $\text{tr} \phi < -2$, then $\pi \cong E_2^+$; if $\text{det} \phi = -1$, then $\pi \cong E_2^-$ (see Section 7.2).

Next, we consider a torus semi-bundle $N_\phi$ where

$$\phi = B = \begin{bmatrix} p & q \\ u & v \end{bmatrix} \in \text{GL}(2, \mathbb{Z})$$

and $pvqu \neq 0$ (i.e., not Seifert-fibered). Notice that the torus semi-bundle $N_\phi = N \cup_B N$ is possibly non-orientable. Recall $\pi_1(N) = \langle a, b \mid a\beta a^{-1} = \beta^{-1} \rangle$ and $\pi_1(\partial N) \cong \mathbb{Z}^2$ is an index 2 subgroup of $\pi_1(N)$ generated by $a^2$ and $\beta$. Then $\phi : \partial N \to \partial N$ is a homomorphism that induces a homomorphism $\phi_* : \pi_1(\partial N) \to \pi_1(\partial N)$ so that $\phi_*((\alpha^2) = p^2a^2 + q\beta$, $\phi_*((\beta) = u\alpha a^2 + v\beta$. So, $pv - qu = \pm 1$. Now by Seifert-van Kampen’s theorem, the fundamental group $\pi$ of $N_\phi$ has the following presentation:

$$\pi = \langle x, y, z, w \mid xyx = y^{-1}, z^2 = x^2z^2, w = x^2w, zwz = w^{-1} \rangle.$$
Using the fact that $pqvu \neq 0$, it can be seen that $|\text{tr}A| > 2$. Notice also that $[p, q]^i$ belongs to $\ker (I - M)$, and $[p \mp (pv + qu), q \mp 2qv]^i = [p, q]^i = \mp [pv + qu, 2qv]^i$ belongs to $\ker (A - M)$. Since $\det (I - M) = 0$ and $I - M \neq 0$, we have that $\ker (I - M)$ is one-dimensional, and since $\gcd (p, q) = 1$ it follows that $[p, q]^i$ is a generator of $\ker (I - M)$. By Lemma 7.2, we can deduce also that $\text{im} (I + M)$ has index 2 in $\ker (I - M)$.

A simple calculation shows that $\gcd (pv + qu, 2qv) = 1$ and it follows that $[pv + qu, 2qv]^i$ is a generator of $\ker (A - M)$. Moreover, $\text{im} (A^{-1} + M)$ has index 2 in $\ker (A - M)$. Finally we notice that $p \mp (pv + qu) \neq 0$ and $q \mp 2qv \neq 0$.

Consequently, if $pv - qu = 1$ and $pv + qu > 0$, or if $pv - qu = -1$ and $pv + qu < 0$,

then $\pi \cong E_3 \left( \begin{bmatrix} p \\ q \end{bmatrix}, \begin{bmatrix} p \mp (pv + qu) \\ q \mp 2qv \end{bmatrix} \right)$ with $t = \tau$, $A = A$ and $M = M$,

if $pv - qu = 1$ and $pv + qu < 0$, or if $pv - qu = -1$ and $pv + qu > 0$,

then $\pi \cong E_9 \left( \begin{bmatrix} p \\ q \end{bmatrix}, \begin{bmatrix} p \mp (pv + qu) \\ q \mp 2qv \end{bmatrix} \right)$ with $s = \tau$, $N = A$ and $M = M$.

Consequently, $E_3$ and $E_9$ are isomorphic to the fundamental group of the spaces which are the union of two twisted $T$-bundles over the Klein bottle.

Recall from [19, Theorem 5.3] the following: A closed 3-manifold has Sol-geometry if and only if it is finitely covered by a torus bundle over $S^3$ with hyperbolic gluing map. In particular, it is either a bundle over $S^1$ with fiber the torus or Klein bottle or is the union of two twisted $I$-bundles over the torus or Klein bottle. Upon this result, we would like to add a few comments.

All the SB-groups $E$ are solvable: $E_0$ and $E_2^\pm$ are 2-step, while $E_3$ and $E_9$ are 3-step. $E_0 \subset E_3$ and $E_2^+ \subset E_9$ of index 2. $E_0$ and $E_2^+$ are extensions of $\mathbb{Z}^2$ by $\mathbb{Z}$: $0 \to \mathbb{Z}^2 \to E \to \mathbb{Z} \to 0$. It is clear that the quotient space $E_2^+ / \text{Sol}$ is a torus bundle over the circle.

Every element of $E_2^-$ can be written uniquely in the form $a^n b^m c^p$. If $E_2^-$ has a Klein bottle subgroup, there are nontrivial elements $a^n b^m c^p$ and $a^n b^m c^p$ such that $(a^n b^m c^p)^{-1} = (a^n b^m c^p)^{-1} \Rightarrow q' = 0$, $N'^{(p')} = -p'$. As $N_2^2 = A$, $N_2$ has irrational eigenvalues. This implies that $p' = 0$. Consequently, $E_2^-$ cannot have a Klein bottle subgroup. Recalling that $E_2^-$ has an index 2 subgroup $\Gamma_A$, the quotient space $E_2^- \setminus \text{Sol}$ is doubly covered by the quotient space $\Gamma_A \setminus \text{Sol}$, which is a torus bundle over the circle. Now, the quotient group $E_2^- / \Gamma_A \equiv \Phi$ acts freely on the torus bundle over the circle: it acts on the base by the rotation by $\pi$ and it acts on the fiber, by a reflection. Hence the quotient space $E_2^- / \text{Sol}$ is again a torus bundle over the circle.

There is no Klein bottle bundle over the circle in Sol-geometry. Assume on the contrary that there exists such one $M$ with fundamental group $\Pi$. Then it is doubly covered by a torus bundle $M_0$ over $S^1$ with hyperbolic gluing map $\phi \in \text{GL}(2, \mathbb{Z})$, and $\Phi$ fits in an exact sequence $1 \to K \to \Pi \to \mathbb{Z} \to 0$, where $K$ is the Klein bottle group. The fundamental group $\Pi_0$ of $M_0$ is an index 2 subgroup of $\Pi$. This induces the following commutative diagram of exact sequences:

$$
\begin{array}{ccccccccc}
1 & 1 & 1 \\
\uparrow & & \uparrow & & \uparrow \\
1 & \to & K/K \cap \Pi_0 & \to & \mathbb{Z}_2 & \to & \mathbb{Z} / (\Pi / K \cap \Pi_0) & \to & 1 \\
\uparrow & & \uparrow & & \uparrow \\
1 & \to & K & \to & \Pi & \to & \mathbb{Z} & \to & 1 \\
\uparrow & & \uparrow & & \uparrow \\
1 & \to & K \cap \Pi_0 & \to & \Pi_0 & \to & \Pi_0 / K \cap \Pi_0 & \to & 1 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
1 & & 1 & & 1
\end{array}
$$
Since $\Pi_0 \cong \mathbb{Z}^2 \times_o \mathbb{Z}$ (as we already know about torus bundles over the circle with hyperbolic gluing maps), i.e., since $\Pi_0 \cong E_0$ or $E^2_1$, it cannot contain a Klein bottle subgroup. Hence $K / K \cap \Pi_0 \cong \mathbb{Z}^2$, $\mathbb{Z} / (\Pi / K \cap \Pi_0) = \{1\}$, $K \cap \Pi_0 = \mathbb{Z}^2$ and $\Pi_0 / K \cap \Pi_0 = \mathbb{Z}$. The bottom exact sequence is $1 \rightarrow \mathbb{Z}^2 \rightarrow \Pi_0 \rightarrow \mathbb{Z} \rightarrow 1$. Remark that $\mathbb{Z}^2$ is the discrete nil-radical of $K$. In fact, it is the nil-radical of $\Pi$. Thus we have the following commutative diagram

$$
\begin{array}{ccc}
1 & \rightarrow & \mathbb{Z}_2 \\
& & |
\downarrow \\
& & \mathbb{Z} \\
1 & \rightarrow & K \\
& & |
\downarrow \\
& & \Pi \\
& & |
\downarrow \\
& & \mathbb{Z}^2 \\
& & |
\downarrow \\
& & \mathbb{Z}^2 \\
\end{array}
$$

The group $Q$ is an extension of $\mathbb{Z}_2$ by $\mathbb{Z}$. Thus it is a semi-direct product $\mathbb{Z}_2 \rtimes \mathbb{Z}$. But the only action of $\mathbb{Z}$ by $\mathbb{Z}_2$ (i.e., the homomorphism $\mathbb{Z} \rightarrow \text{Aut}(\mathbb{Z}_2)$) is trivial. Thus $Q = \mathbb{Z}_2 \times \mathbb{Z}$. Let $a_1, a_2, t, \gamma$ be generating set for $\Pi$, where $\mathbb{Z}^2 = \langle a_1, a_2 \rangle, K = \langle a_1, a_2, \gamma \rangle$ and $Q$ is generated by the images of $\gamma, t$ so that $Q = \langle \tilde{\gamma}, \tilde{t} \rangle = \mathbb{Z}_2 \times \mathbb{Z}$: $\tilde{\gamma}^2 = 1$ and $\tilde{t}^2 = \tilde{t}$. Now, $t$ acts on the nil-radical $\mathbb{Z}^2$ by $\phi$ and we may assume that $\gamma$ acts on $\mathbb{Z}^2$ as, if it is necessary we choose a new set of generators,

$$M = \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ not } M' = \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$ 

If $M'$ is the case, $K$ cannot be a Klein bottle group. If $M$ is the case, then we know that $\gamma$ cannot normalize $\Pi_0 = \langle a_1, a_2, t \rangle$, because $\sqrt{\text{trace}^2 - 4}$ is not an integer. In fact, when $\phi = A$, then by Remark 5.6 $P^{-1}MP$ cannot be an integer matrix. When $\phi \neq A$, then $\phi = N$ is a square root of $A$, and by the same reason as before $P^{-1}MP$ cannot be an integer matrix.

The union $M$ of two twisted $I$-bundles over the torus is double covered by $S^1 \times S^1 \times S^1$ and so is not a Sol manifold (see for example [1, p. 282]). Let $T \times I$ be the twisted $I$-bundle over the torus $T$. Then $\pi_1(T \times I) \cong \mathbb{Z}^2$ is an index 2 subgroup of $\pi_1(T \times I) = \langle \alpha, \beta | [\alpha, \beta] = 1 \rangle$ generated by $\alpha^2$ and $\beta$ as $T \times I$ is the product space of the Möbius band with the circle. By the Seifert-van Kampen’s theorem, the fundamental group $\pi$ of $M$ has the following presentation:

$$\pi = \{x, y, z, w | [x, y] = 1, z^2 = x^2 y^2, w = x^{2u} y^2, [z, w] = 1\}$$

where $p v - q u = \pm 1$. We take

$$a_1 = x^2, a_2 = y, a_3 = x^{-1} z, \alpha = z.$$ 

Since $\pi$ is generated by $x, y$ and $z$, it is also generated by $a_1, a_2, \tau$ and $\alpha$, and

$$a_1 a_2 a_1^{-1} a_2^{-1} = 1,$$

$$a^2 = a^u a^v,$$

$$\alpha(a^u a^v) \alpha^{-1} = a^u a^v,$$

$$a_1 a_2 a_1^{-1} a_2^{-1} = 1,$$

$$a_1 a_2 a_1^{-1} a_2^{-1} = 1.$$
Hence

\[ \pi = \langle a_1, a_2, a_3, \alpha \mid [a_i, a_j] = 1, a_i^2 = a_i^q a_j^q, \]
\[ a a_i a_i^{-1} = a_i, a a_2 a_2^{-1} = a_2, a a_3 a_3^{-1} = a_i^{(q-1)/2} a_j^{(q-1)/2} \rangle. \]

Consequently, \( \pi \) fits in a short exact sequence \( 1 \rightarrow \mathbb{Z}^3 \rightarrow \pi \rightarrow \mathbb{Z}_2 \rightarrow 1 \) where \( \mathbb{Z}_2 \) acts on \( \mathbb{Z}^3 \) by the matrix

\[
\begin{pmatrix}
1 & 0 & * \\
0 & 1 & * \\
0 & 0 & -1
\end{pmatrix}.
\]

This implies that \( \pi \) is isomorphic to a 3-dimensional non-orientable Bieberbach group with holonomy group isomorphic to \( \mathbb{Z}_2 \). Namely, \( \pi \) is isomorphic to either \( B_1 \) or \( B_2 \) (see for example [22, Theorem 3.5.9]). Indeed, if \( q \) is even, then as \( pq - qu = \pm 1 \), \( p \) must be odd, and \( \pi \) is generated by \( \{ t_1 = a_i a_j, t_2 = a_i^q a_j^q, t_3 = a_i^{-(q-1)/2} a_j^{-(q-1)/2} \} \) where

\[ [t_i, t_j] = 1, \quad \epsilon - 1 = t_1, \quad \epsilon t_2 \epsilon^{-1} = t_2, \quad \epsilon t_3 \epsilon^{-1} = t_3^{-1}. \]

Thus \( \pi \) is isomorphic to \( B_1 \). If \( q \) is odd, then \( \pi \) is generated by \( \{ t_1 = a_i^{-1}, t_2 = a_i^q a_j^q, t_3 = a_i^{-(q-1)/2} a_j^{-(q-1)/2} \} \) where

\[ [t_i, t_j] = 1, \quad \epsilon - 1 = t_1, \quad \epsilon t_2 \epsilon^{-1} = t_2, \quad \epsilon t_3 \epsilon^{-1} = t_1 t_2 t_3^{-1}. \]

Thus \( \pi \) is isomorphic to \( B_2 \).

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References