NIELSEN SPECTRUM OF MAPS ON INFRA-SOLVMANIFOLDS MODELED ON $\text{Sol}_0^4$

Jong Bum Lee

Abstract. The 4-dimensional solvable Lie group $\text{Sol}_0^4$ does not admit a lattice. The purpose of this paper is two-fold. We study poly-crystallographic groups of $\text{Sol}_0^4$, and then we study Nielsen fixed point theory on the spaces modeled on $\text{Sol}_0^4$.

1. Introduction

It is well known that every connected, simply connected nilpotent Lie group admits a lattice, i.e., a discrete cocompact subgroup. The famous Bieberbach’s first theorem is available for nilpotent Lie groups as proved by Auslander, see [22]. Hence finding almost crystallographic groups and almost Bieberbach groups is a great concern, because such groups give rise to infra-nilorbifolds and infra-nilmanifolds, and infra-nilmanifolds are almost flat manifolds.

The study of Nielsen fixed point theory on infra-nilmanifolds has been successful because such manifolds allow finite regular coverings by nilmanifolds, and as a result, one can use the averaging formula for Nielsen numbers, [1, 7, 14–16, 19, 20]. Certain solvable Lie groups of type (R) (completely solvable, or supersolvable) allow lattices. The 3-dimensional solvable Lie group $\text{Sol}$ is such an example. If this is the case, one can extend the averaging formula for Nielsen numbers and use it to study Nielsen fixed point theory on infra-solvmanifolds of type (R), [2–6, 8, 12, 13, 17, 21].

Among 4-dimensional solvable Lie groups, only $\text{Sol}_0^4$ and $\text{Sol}_0^4$ do not have a lattice. Hence we cannot study their crystallographic groups nor Bieberbach groups. $\text{Sol}_0^4$ does not have a compact-form, yet $\text{Sol}_0^4$ has infinitely many compact-forms, see [18]. In particular, the averaging formula is not applicable.

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The purpose of this paper is to study compact-forms of $\text{Sol}_0^4$, and then to study Nielsen fixed point theory on compact spaces modeled on $\text{Sol}_0^4$.

2. Poly-crystallographic groups of $\text{Sol}_0^4$

2.1. The Lie group $\text{Sol}_0^4$

Recall, for example from [18], that $\text{Sol}_0^4 = \mathbb{R}^3 \times_{\psi} \mathbb{R}$ where

$$\psi(s) = \begin{bmatrix} e^s & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^{-2s} \end{bmatrix}.$$  

Then it can be embedded in $\text{Aff}(4)$ as

$$\left\{ \begin{bmatrix} \psi(s) & 0 \\ 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right\} \subset \text{Aff}(4) \subset \text{GL}(5, \mathbb{R}),$$

where $\psi(s) \in \text{GL}(3, \mathbb{R})$, $s \in \mathbb{R}$ and $x \in \mathbb{R}^3$ is a column vector. Furthermore, it can be seen that

$$\text{Aut}(\text{Sol}_0^4) = \text{GL}(2, \mathbb{R}) \times \text{GL}(1, \mathbb{R}),$$

which is generated by

$$\begin{bmatrix} p_{11} & p_{12} & 0 & 0 & 0 \\ p_{21} & p_{22} & 0 & 0 & 0 \\ 0 & 0 & p_{33} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$  

A maximal compact subgroup is $\text{O}(2) \times \text{O}(1)$, and its identity component is $\text{SO}(2)$.

Recall also that $\text{Sol}_0^4$ does not admit a discrete cocompact subgroup. So, this is a type (R) (i.e., supersolvable) counter-example to the generalized Bieberbach’s first theorem.

Consequently, we shall be interested in classifying discrete cocompact subgroups of $\text{Sol}_0^4 \rtimes (\text{O}(2) \times \text{O}(1))$. We call such a group a poly-crystallographic group of $\text{Sol}_0^4$, see [26, Theorem 3] for details of poly-crystallographic groups.

2.2. Poly-crystallographic groups of $\text{Sol}_0^4$

Let $G := \text{Sol}_0^4 \rtimes \text{SO}(2) = (\mathbb{R}^3 \times_{\psi} \mathbb{R}) \rtimes \text{SO}(2)$. Here, $\text{SO}(2)$ acts on $\mathbb{R}^3 \times_{\psi} \mathbb{R}$ as

$$\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \ast (x, s) := \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} (x, s).$$

Write

$$\nu(\theta) = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$
Then $G = \mathbb{R}^3 \rtimes \psi'(\mathbb{R} \times \text{SO}(2))$, where $\psi'$ is determined by $\psi(s)r(\theta) = r(\theta)\psi(s)$.

Let $\Gamma \subset G$ be a poly-crystallographic group of $\text{Sol}_4^3$. The nilradical of $\text{Sol}_4^3 = \mathbb{R}^3 \rtimes_\psi \mathbb{R}$ is $\mathbb{R}^3$, and $G/\mathbb{R}^3 = \mathbb{R} \times \text{SO}(2)$. Consider the commutative diagram between short exact sequences:

$$
\begin{array}{cccc}
1 & \longrightarrow & \mathbb{R}^3 & \longrightarrow & G & \longrightarrow & \mathbb{R} \times \text{SO}(2) & \longrightarrow & 1 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \Gamma \cap \mathbb{R}^3 & \longrightarrow & \Gamma & \longrightarrow & \Gamma/(\Gamma \cap \mathbb{R}^3) & \longrightarrow & 1
\end{array}
$$

By [26, Proposition 5.1], $\Gamma \cap \mathbb{R}^3 = \mathbb{Z}^3$ is a lattice of $\mathbb{R}^3$. Therefore, $\Gamma$ is of the form $\mathbb{Z}^3 \rtimes K$ where $K := \Gamma/(\Gamma \cap \mathbb{R}^3)$ is a discrete cocompact subgroup of $\mathbb{R} \times \text{SO}(2)$.

Next, we will study discrete cocompact subgroups of $\mathbb{R} \times \text{SO}(2)$. Let $K_1$ and $K_2$ denote the projections of $K$ into the two factors, respectively. Then $K$ is a subgroup of $K_1 \times K_2$. In general, a subgroup of a product group need not itself be a product group. We remark that $K_1$ must be a discrete cocompact subgroup of $\mathbb{R}$, and hence $K_1 \cong \mathbb{Z}$. Note also that $K_2$ is a subgroup of $\text{SO}(2)$, and hence it is a finite cyclic group or an infinite cyclic group (which is dense in $\text{SO}(2)$).

If $K_2$ is a finite group then $K' := \text{pr}_2^{-1}(\{1\})$ is a finite index subgroup of $K$ where $\text{pr}_2 : \mathbb{R} \times \text{SO}(2) \to \text{SO}(2)$ is the projection onto the second factor. Then the inverse image $\Gamma'$ of $K'$ under $\Gamma \to K$ is a finite index subgroup of $\Gamma$. We can regard $K'$ as a subgroup of $\mathbb{R}$, and $\Gamma'$ as a subgroup of $\mathbb{R} \rtimes_\psi \mathbb{R}$. However, this is impossible as $\text{Sol}_4^3 = \mathbb{R}^3 \rtimes_\psi \mathbb{R}$ does not admit such a subgroup (a lattice). Therefore $K_2$ must be an infinite cyclic subgroup of $\text{SO}(2)$. That is, $K_2$ is of the form

$$
K_2 = \left\{ R(n\theta_0) = \begin{bmatrix} \cos(n\theta_0) & \sin(n\theta_0) \\ -\sin(n\theta_0) & \cos(n\theta_0) \end{bmatrix} \mid n \in \mathbb{Z} \right\} \subset \text{SO}(2),
$$

where $\theta_0$ is an irrational multiple of $\pi$.

Let $K_i' = K \cap K_i$ for $i = 1, 2$. Then we can see that

$$
K'/(K_1' \times K_2') \cong K_1/K_1' \cong K_2/K_2'.
$$

Indeed, the kernel of the canonical map $K \to K_i/K_i'$ is $K_i' \times K_2'$. We claim that $K_1' = K_2' = \{1\}$. Assume $K_1' \neq \{1\}$. Then the inverse image $\Gamma_1'$ of $K_1'$ under $\Gamma \to K$ sits inside $\text{Sol}_4^3$ and it fits the short exact sequence $1 \to \mathbb{Z}^3 \to \Gamma_1' \to K_1' \to 1$, hence it is a lattice of $\text{Sol}_4^3$ which is impossible. Thus $K_1' = \{1\}$.

Since as a group $K_2 \cong K_1$, we have $K_2' \cong K_1' = \{1\}$.

Consequently, $K \cong \mathbb{Z}$, and $\Gamma \cong \mathbb{Z} \rtimes A \mathbb{Z} \subset \mathbb{R}^3 \rtimes (\mathbb{R} \times \text{SO}(2))$ where

$$
\mathbb{Z} \longrightarrow \mathbb{R} \times \text{SO}(2), \quad n \longmapsto (ns_0, R(n\theta_0))
$$
for some nonzero \( s_0, \theta_0 \in \mathbb{R} \) where \( \theta_0 \) is an irrational multiple of \( \pi \). This implies that there exists a matrix \( P \in GL(3, \mathbb{R}) \) such that

\[
PAP^{-1} = \begin{bmatrix} e^{s_0} & 0 & 0 \\ 0 & e^{s_0} & 0 \\ 0 & 0 & e^{-2s_0} \end{bmatrix} \begin{bmatrix} \cos \theta_0 & \sin \theta_0 & 0 \\ -\sin \theta_0 & \cos \theta_0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \psi(s_0) r(\theta_0).
\]

With \( \varphi : \mathbb{R} \to GL(3, \mathbb{R}), t \mapsto \varphi(t) = \psi(t s_0) r(t \theta_0) \), the \( \mathbb{Z} \)-action on \( \mathbb{R}^3 \) is \( n \ast x = \varphi(n)(x) = P A^n P^{-1}(x) \). The embedding \( \mathbb{Z}^3 \rtimes_A \mathbb{Z} \hookrightarrow \mathbb{R}^3 \rtimes_{s_0} (\mathbb{R} \times SO(2)) \) is given explicitly as

\[
(m, n) \mapsto (P m, (n s_0, R(n \theta_0))).
\]

We remark that \( A \) has a positive real eigenvalue \( a = e^{-2s_0} \) and complex eigenvalues \( e^{s_0 \pm i \theta_0} \). By taking \( -s_0 \) or equivalently by taking \( A^{-1} \) if it is necessary, we may assume that \( a > 1 \).

The converse is known, see for example [10, 25]. If \( A \in SL(3, \mathbb{Z}) \) has positive real eigenvalue \( a \neq 1 \) and complex eigenvalues, then the group \( \mathbb{Z}^3 \rtimes_A \mathbb{Z} \) can be realized as a discrete cocompact subgroup of \( \text{Sol}^4_0 \rtimes SO(2) \). The Lie group \( \text{Sol}^0_3 \rtimes SO(2) \) acts on \( \mathbb{C} \times H \) by isometries and the complex surface \( \Gamma/(\mathbb{C} \times H) \) has finite volume, called an Inoue surface with \( \text{Sol}^4_0 \)-geometry.

It is shown in [18, Theorem 4.3] that there are countably infinite distinct, discrete cocompact subgroups \( \Gamma \cong \mathbb{Z}^3 \rtimes \mathbb{Z} \) in \( \text{Sol}^4_0 \rtimes SO(2) \).

Let \( \Gamma \) be any poly-crystallographic group of \( \text{Sol}^4_0 \). By definition, \( \Gamma \subset \text{Sol}^4_0 \rtimes (O(2) \times O(1)) \). However, by [18, Theorem 4.3] again, we must have \( \Gamma \subset \text{Sol}^0_3 \rtimes SO(2) \). In summary, there are countably infinite many poly-crystallographic groups of \( \text{Sol}^4_0 \), all of them of the form \( \mathbb{Z}^3 \rtimes \mathbb{Z} \). In particular, they are torsion-free.

3. Infra-solvmanifolds modeled on \( \text{Sol}^4_0 \)

Let \( \Gamma = \mathbb{Z}^3 \rtimes_A \mathbb{Z} \) be a poly-crystallographic group with an embedding \( \iota : \Gamma \hookrightarrow \text{Sol}^4_0 \rtimes SO(2) = \mathbb{R}^3 \rtimes_{s_0} (\mathbb{R} \times SO(2)) \) so that its image in \( SO(2) \) is dense. As a result, \( \text{Sol}^4_0 \rtimes SO(2) \) is the Lie hull of \( \iota(\Gamma) \) in \( \text{Sol}^4_0 \rtimes SO(2) \). The orbit space \( M = \Gamma/\text{Sol}^4_0 = \Gamma/(\text{Sol}^4_0 \rtimes SO(2))/SO(2) \) is an infra-solvmanifold modeled on the supersolvable Lie group \( \text{Sol}^4_0 \).

In this section, we go one step further to examine such infra-solvmanifolds closely. Similarly, infra-solvmanifolds modeled on \( \text{Sol}^4_1 \) were studied in [23, 24]. Unlike \( \text{Sol}^4_0 \), \( \text{Sol}^4_1 \) admits a lattice.

Regarding \( \iota \) as inclusion, we have \( \Gamma \cap \text{Sol}^4_0 = \mathbb{Z}^3 \), and its Lie hull in \( \text{Sol}^4_0 \) and hence in \( \text{Sol}^4_0 \rtimes SO(2) \) is \( \mathbb{R}^3 \), which is the nilradical of \( \text{Sol}^4_0 \). Consider the following commutative diagram between short exact sequences:

\[
\begin{array}{cccccccc}
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \mathbb{R}^3 & \longrightarrow & \text{Sol}^4_0 \rtimes SO(2) & \longrightarrow & \mathbb{R} \times SO(2) & \longrightarrow & 1 \\
1 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \Gamma & \longrightarrow & \mathbb{Z} & \longrightarrow & 1
\end{array}
\]
where the rightmost vertical map is given by $n \mapsto (ns_0, \mathcal{R}(n\theta_0))$, and hence $\mathbb{Z}$ acts on $\mathbb{R}^3$ via $\varphi(1) = PAP^{-1}$ of Section 2. Furthermore, since $\Gamma = \mathbb{Z}^3 \rtimes_A \mathbb{Z}$, the universal covering projection $\text{Sol}_0^4 \to M$ has a decomposition by covering projections

$$\text{Sol}_0^4 \to \tilde{M} := \mathbb{Z}^3/\text{Sol}_0^4 \xrightarrow{\mathbb{Z}_3} M = \Gamma/\text{Sol}_0^4.$$

On the other hand, the space $\tilde{M}$ fits the fibration

$$
\mathbb{Z}^3/\mathbb{R}^3 \to \tilde{M} = \mathbb{Z}^3/(\mathbb{R}^3 \rtimes \mathbb{Z}) \to \mathbb{R}
$$

with fiber $T^3$ and base $\mathbb{R}$. The action by the transformation group $\mathbb{Z}$ on the orbit space $\tilde{M}$ is fiber preserving. Consequently, we have the fibration

$$
\mathbb{Z}^3/\mathbb{R}^3 \to \mathbb{Z}\tilde{M} = M \to \mathbb{Z}\mathbb{R}
$$

with fiber $T^3$ and base $S^1$. This implies that the infra-solvmanifold $M = \Gamma/\text{Sol}_0^4$ is the mapping torus $M_A$ of the diffeomorphism $\varphi(1) = PAP^{-1} : T^3 \to T^3$. Clearly it is diffeomorphic to the mapping torus $M_A$ of the diffeomorphism $A : T^3 \to T^3$.

Conversely a mapping torus $M_A$ of $A \in \text{GL}(3, \mathbb{Z})$ has infra-solvmanifold structure modeled on $\text{Sol}_0^4$ if $A \in \text{SL}(3, \mathbb{Z})$ has a positive real eigenvalue and two complex eigenvalues. We refer to [25] for details.

4. Nielsen spectrum of maps on the mapping torus $M_A$

The infra-solvmanifold $\Gamma/\text{Sol}_0^4$ with fundamental group $\Gamma = \mathbb{Z}^3 \rtimes_A \mathbb{Z}$ is diffeomorphic to the mapping torus $M_A$. In this section, we shall study the set of Reidemeister numbers of all endomorphisms on $\Gamma$, and the sets of Lefschetz numbers and Nielsen numbers of all self-maps on $M_A$.

Lemma 4.1. If $A \in \text{GL}(n, \mathbb{Z})$ does not have an eigenvalue 1, then the subgroup $\mathbb{Z}^n$ of the group $\Gamma = \mathbb{Z}^n \rtimes_A \mathbb{Z}$ is a fully invariant subgroup.

Proof. Since the quotient group $\Gamma/\mathbb{Z}^n = \mathbb{Z}$ is abelian, we have $[\Gamma, \Gamma] \subset \mathbb{Z}^n$. Since the group $\Gamma$ can be presented as

$$\Gamma = \langle x_1, \ldots, x_n, t \mid [x_i, x_j] = 1, \ tx_it^{-1} = \theta(x_i) \rangle,$$

it follows that $[\Gamma, \Gamma]$ is the image of the homomorphism $I - A : \mathbb{Z}^n \to \mathbb{Z}^n$. Hence the index of $[\Gamma, \Gamma]$ in $\mathbb{Z}^n$ equals $|\det(I - A)|$. By the assumption on $A$, $\det(I - A) \neq 0$. Thus, $[\Gamma, \Gamma]$ has finite index in $\mathbb{Z}^n$.

Now we shall show that $\mathbb{Z}^n$ is a fully invariant subgroup of $\Gamma$. Let $\phi$ be an endomorphism of $\Gamma$ and let $m \in \mathbb{Z}^n$. Then $\phi(m) = (m', k) \in \mathbb{Z}^n \rtimes_\theta \mathbb{Z}$ for some $m' \in \mathbb{Z}^n$ and $k \in \mathbb{Z}$. Since $[\Gamma, \Gamma]$ has finite index in $\mathbb{Z}^n$, $m' \in [\Gamma, \Gamma]$ for some $\ell \in \mathbb{Z}$, and since $[\Gamma, \Gamma]$ is a fully invariant subgroup of $\Gamma$, $\phi(m') \in [\Gamma, \Gamma]$.

Observe that

$$\phi(m') = \phi(m')^\ell = (m', k)^\ell = (m'', k\ell) \in [\Gamma, \Gamma]$$

for some $m'' \in \mathbb{Z}^n$. This implies that $k = 0$, hence $\phi(m) = (m', 0) \in \mathbb{Z}^n$. Consequently, $\mathbb{Z}^n$ is a fully invariant subgroup of $\Gamma$. □
Let \( f : M_A \to M_A \) be a self-map, inducing an endomorphism \( \phi : \Gamma \to \Gamma \) on the group \( \Gamma \) of covering transformations of the universal cover \( \text{Sol}_4^0 \to M_A \).

By Lemma 4.1, \( \Gamma \cap \text{Sol}_4^0 = Z_3 \) is a fully invariant subgroup of \( \Gamma \), and hence we obtain the following commutative diagram:

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & Z^3 & \longrightarrow & \Gamma & \longrightarrow & Z & \longrightarrow & 1 \\
& & \phi' & \downarrow \phi & \downarrow \phi' & & \downarrow \bar{\phi} & & \\
1 & \longrightarrow & Z^3 & \longrightarrow & \Gamma & \longrightarrow & Z & \longrightarrow & 1
\end{array}
\]

This commutative diagram induces the identity

\[
\phi' \circ \varphi(1) = \varphi(\bar{\phi}(1)) \circ \phi'.
\]

With a suitable basis for \( Z^3 \), if \( B \) is the matrix of \( \phi' \) and \( \bar{\phi}(1) = k \), then the above identity means that

\[(4.1) \quad A^k B = BA.\]

Moreover, from the commutative diagram (4.1) we may assume that \( f \) is a fiber-preserving map so that the following diagram is commutative:

\[
\begin{array}{ccccccccc}
T^3 & \longrightarrow & M_A & \longrightarrow & S^1 \\
\downarrow f' & & \downarrow f & & \downarrow f \\
T^3 & \longrightarrow & M_A & \longrightarrow & S^1
\end{array}
\]

4.1. Converse of (4.1)

Given endomorphisms \( \phi' : Z^3 \to Z^3 \) and \( \bar{\phi} : Z \to Z \), we want to know if there is an endomorphism \( \phi : \Gamma \to \Gamma \) fitting the commutative diagram (4.1). Assume (4.1) is given. Then

\[
\phi(m, n) = \phi((m, 0)(0, n)) = (\phi'(m), 0)(\xi(n), \bar{\phi}(n)) = (\phi'(m) + \xi(n), \bar{\phi}(n))
\]

for \((m, n) \in \Gamma\), where \( \xi : Z \to Z^3 \). Since \( \phi \) is an endomorphism, one can show that \( \xi \) is a crossed homomorphism, i.e.,

\[
(4.5) \quad \xi(m + n) = \xi(m) + \theta(\bar{\phi}(m))\xi(n) = \xi(m) + A^kn\xi(n).
\]

Such a crossed homomorphism is determined by the image \( \xi(1) \).

In summary, we see that with \( \phi' \) and \( \bar{\phi} \), simply by choosing \( \xi(1) \in Z^3 \) we can define an endomorphism \( \phi \) so that the commutative diagram (4.1) is obtained.
4.2. Analysis of (4.2)

Recall

\[
PAP^{-1} = \begin{bmatrix}
e^{s_0} & 0 & 0 \\
0 & e^{s_0} & 0 \\
0 & 0 & e^{-2s_0}
\end{bmatrix} \begin{bmatrix}
\cos \theta_0 & \sin \theta_0 & 0 \\
-\sin \theta_0 & \cos \theta_0 & 0 \\
0 & 0 & 1
\end{bmatrix} =: \begin{bmatrix}
E & 0 & 0 \\
0 & 0 & e^{-2s_0}
\end{bmatrix} \begin{bmatrix}
R & 0 \\
0 & 1
\end{bmatrix}
\]

and let

\[
PBP^{-1} = \begin{bmatrix}
a & b & x \\
c & d & y \\
u & v & w
\end{bmatrix} =: \begin{bmatrix}
B & x \\
u & w
\end{bmatrix},
\]

where \(x, u\) are 2-dimensional column vector and row vector respectively. If \(k = 1\), the identity (4.2) yields that

\[
PBP^{-1} = \begin{bmatrix}
a & b \\
-\bar{b} & a \\
0 & 0
\end{bmatrix} \begin{bmatrix}
w
\end{bmatrix}.
\]

Assume \(k \neq 1\). Denoting \(N := ER\), by (4.2) we have \(N^k B = BN\). Since \(N\) has complex eigenvalues \(e^{s_0 \pm i\theta_0}\), it can be diagonalized with its eigenvalues on the diagonal entries. It follows that \(B = 0\) since the modulus of our eigenvalues are not roots of unity. We can show further that \(B\) itself is the zero matrix.

4.3. The Reidemeister number \(R(f)\) of \(f\)

First we determine the Reidemeister number \(R(f)\) of \(f\). By definition, \(R(f)\) is just the Reidemeister number \(R(\phi)\) of the endomorphism \(\phi\), which is the cardinality of the Reidemeister set \(\mathcal{R}[\phi]\).

Recall from [6, Sect.1] or [9, Lemma 3.1] that the commutative diagram (4.1) induces a 6-term exact sequence between fixed point groups and Reidemeister sets:

\[
1 \longrightarrow \text{fix}(\phi') \longrightarrow \text{fix}(\phi) \longrightarrow \text{fix}(\bar{\phi}) \longrightarrow \mathcal{R}[\phi'] \longrightarrow \mathcal{R}[\phi] \longrightarrow \mathcal{R}[\bar{\phi}] \longrightarrow 1.
\]

It is clear that \(\text{fix}(\bar{\phi}) = \mathbb{Z}\) if \(k = 1\), and \(\text{fix}(\bar{\phi}) = \{0\}\) if \(k \neq 1\). It is well-known that \(R(\bar{\phi}) = \sigma(1 - k)\) and \(R(\phi') = \sigma(\det(I - M))\), where \(\sigma : \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}\) is defined by \(\sigma(0) = \infty\) and \(\sigma(x) = |x|\) for all \(x \neq 0\).

It is immediate that if \(k = 1\), then \(R(\bar{\phi}) = \infty\), and hence from the exact sequence we have \(R(\phi) = \infty\). In particular, we have

\[
R(\phi) = \infty = R(\phi') \cdot R(\bar{\phi}).
\]

On the other hand, if \(k \neq 1\) we have a short exact sequence of sets \(1 \rightarrow \mathcal{R}[\phi'] \rightarrow \mathcal{R}[\phi] \rightarrow \mathcal{R}[\bar{\phi}] \rightarrow 1\) with the finite quotient set \(\mathcal{R}[\bar{\phi}]\) of \(|1 - k|\) elements, and

\[
R(\phi) = R(\phi') \cdot R(\bar{\phi}) = |1 - k| \cdot R(\phi'),
\]

hence \(R(\phi) = \infty\) if and only if \(R(\phi') = \infty\) if and only if \(\det(I - M) = 0\) if and only if \(M\) has an eigenvalue 1.
Proposition 4.2. Let \( f : M \to M \) be a self-map on an infra-solvmanifold \( M = \Gamma \backslash \text{Sol}_4 \) modeled on \( \text{Sol}_4 \). Then the Reidemeister number of \( f \) is
\[
R(f) = \sigma(1 - k).
\]
In particular, \( R(f) = \infty \) if and only if \( k = 1 \). The Reidemeister spectrum of \( M \) is \( \text{Spec}_R(M) = \mathbb{N} \cup \{ \infty \} \).

4.4. The Nielsen number \( N(f) \) of \( f \)

Because the supersolvable Lie group \( \text{Sol}_4 \) does not admit a lattice, the infra-solvmanifold \( M_A = \Gamma \backslash \text{Sol}_4 \) cannot be finitely covered by a solvmanifold. Therefore, in calculating the Nielsen numbers of self-maps on \( M_A \), one cannot use the averaging formula for Nielsen numbers, [17]. However, since \( M_A \) has fibration structure, we may use the fibration technique employed in Jiang’s book, [11].

A fibration \( E \to B \) with fiber \( F \) is orientable if for every element \( [\omega] \in \pi_1(B) \) the induced map \( \tau_{[\omega]} : F \to F \) on \( F \) is homotopic to the identity. We note that our fiber bundle \( M_A \) over \( S^1 \) with fiber \( T^3 \) is non-orientable. The action of \( \pi_1(S^1) \) on the fiber \( T^3 \) induces an action on \( \mathbb{Z}^3 = \pi_1(T^3) \) which can be represented by conjugation of \( \mathbb{Z}^3 \) by elements of \( Z = \pi_1(S^1) \). Since the induced \( Z \)-action on \( \mathbb{Z}^3 \) is given by \( A \), it shows that our fibration is not orientable.

Assume we are given the commutative diagram (4.3). Note that \( N(\bar{f}) = |L(\bar{f})| = |1 - k| \). If \( k = 1 \), then \( \bar{f} \) and hence \( f \) are homotopic to fixed point free maps, hence \( N(f) = 0 \). That is, every fixed point class of \( f \) has index 0, hence \( L(f) = 0 \).

Now we shall consider the case where \( k \neq 1 \). We may assume \( \bar{f} : S^1 \to S^1 \) is given by \( z \mapsto z^k \). The set of fixed points of \( \bar{f} \) is
\[
\text{Fix}(\bar{f}) = \{z_\ell := e^{2\pi i \frac{\ell}{k}} | \ell = 0, 1, \ldots, |k - 2| \},
\]
and every fixed point class of \( \bar{f} \) contains a single fixed point, each of the same index \( \pm 1 \). Henceforth, we will identify the set of fixed point classes of \( f \), \( \text{FPC}(f) \), with \( \text{Fix}(\bar{f}) \).

For \( z_\ell = z_0^\ell \in \text{Fix}(f) \), let \( f_\ell : T^3_L \to T^3_L \) be the restriction of \( f : M_A \to M_A \) to the fiber \( T^3_L \) over \( z_\ell \). There is a sequence of maps:
\[
\text{FPC}(f_\ell) \xrightarrow{\text{FPC}} \text{FPC}(f) \xrightarrow{\pi_{\text{FPC}}} \text{FPC}(\bar{f}).
\]

Let \( \tilde{f} : \text{Sol}_4 \to \text{Sol}_4 \) be a (fixed) lifting of \( f \). Then every lifting of \( f \) is of the form \( \gamma \circ \tilde{f} \) for some covering transformation \( \gamma \in \Gamma \). Recall that every fixed point class of \( f \) is of the form \( F_\gamma = \pi(\text{Fix}(\gamma \circ \tilde{f})) \). For \( \gamma \in \Gamma \), we denote by \( \mu(\gamma) \) the conjugation by \( \gamma \), and by \( \bar{\gamma} \in \mathbb{Z} \) the image of \( \gamma \) under the projection.
\[ \Gamma \to \mathbb{Z} \]. From the commutative diagram (4.1), we have a commutative diagram:

\[
\begin{array}{cccc}
1 & \to & \mathbb{Z}^3 & \to & \Gamma & \to & \mathbb{Z} & \to & 1 \\
& & \mu(\gamma) \circ \phi' & & \mu(\gamma) \circ \phi & & \mu(\gamma) \circ \bar{\phi} & & \\
1 & \to & \mathbb{Z}^3 & \to & \Gamma & \to & \mathbb{Z} & \to & 1
\end{array}
\]

(4.6)

By [11, Proposition III.1.5], we know that
\[ \#i_{\text{FPC}}(F_{\gamma}) = [\text{fix}(\mu(\bar{\gamma}) \circ \bar{\phi}) : \pi(\text{fix}(\mu(\gamma) \circ \phi))]. \]

In our case, \( \mu(\bar{\gamma}) \circ \bar{\phi} = \bar{\phi} \) is the multiplication by the integer \( k \neq 1 \). Hence \( \text{fix}(\bar{\phi}) = \{0\} \). This proves that \( i_{\text{FPC}} : \text{FPC}(f_{\ell}) \to \text{FPC}(f) \) is injective for all \( z_{\ell} \in \text{Fix}(f) \). Moreover, we can see that \( \pi_{\text{FPC}} : \text{FPC}(f) \to \text{FPC}(\bar{f}) = \text{Fix}(f) \) is surjective. For any \( z_{\ell} \in \text{Fix}(f) \), when \( k \neq 1 \), we previously observed that the matrix induced by \( f_{\ell} \) is a zero matrix. This implies that \( f_{\ell} \) is homotopic to a constant map, hence has a unique (essential) fixed point class \( F'_{\ell} \), so \( L(f_{\ell}) = N(f_{\ell}) = 1 \) is independent of \( z_{\ell} \). This yields a fixed point class \( F_{\ell} \) of \( f \) which is mapped to the fixed point class \( z_{\ell} \). Furthermore,

\[ \text{index}(F_{\ell}) = \text{index}(z_{\ell}) \cdot \text{index}(F'_{\ell}). \]

In summary, we have shown that when \( k \neq 1 \),

\[ \text{FPC}(f) = \bigcup_{z_{\ell} \in \text{Fix}(f)} \text{FPC}(f_{\ell}). \]

Furthermore, we have

\[ L(f) = \sum_{z_{\ell} \in \text{Fix}(f)} \text{index}(F_{\ell}) = \sum_{z_{\ell} \in \text{Fix}(f)} \text{index}(z_{\ell}) = L(f), \]

\[ N(f) = \sum_{z_{\ell} \in \text{Fix}(f)} N(f_{\ell}) = \sum_{z_{\ell} \in \text{Fix}(f)} 1 = N(f), \]

\[ |L(f)| = N(f). \]

**Theorem 4.3.** Let \( f : M \to M \) be a self-map on an infra-solvmanifold \( M = \Gamma \backslash \text{Sol}_0^4 \) modeled on \( \text{Sol}_0^4 \). Then the Lefschetz number and the Nielsen number of \( f \) are

\[ L(f) = 1 - k, \quad N(f) = |1 - k|. \]

In particular, \( L(f) = 0 \) if and only if \( N(f) = 0 \) if and only if \( k = 1 \). The Lefschetz spectrum and the Nielsen spectrum of \( M \) are \( \text{Spec}_L(M) = \mathbb{Z} \) and \( \text{Spec}_N(M) = \mathbb{N} \cup \{0\} \).

**References**


Jong Bum Lee

Department of Mathematics

Sogang University

Seoul 04107, Korea

Email address: jlee@sogang.ac.kr