Solution
12.3~14.5

12.3

42. $|\mathbf{a}| = \sqrt{1+16+64} = \sqrt{81} = 9$ so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\text{comp}_{\mathbf{a}}\,\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{1}{9}(-12+4+16) = \frac{8}{9}$, while the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\text{proj}_{\mathbf{a}}\,\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = \frac{8}{9}\cdot\frac{1}{9}\langle -1,4,8\rangle = \frac{8}{81}\langle -1,4,8\rangle = \left\langle -\frac{8}{81}, \frac{32}{81}, \frac{64}{81}\right\rangle$.

44. $|\mathbf{a}| = \sqrt{1+4+9} = \sqrt{14}$ so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\text{comp}_{\mathbf{a}}\,\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \dfrac{5+0-3}{\sqrt{14}} = \dfrac{2}{\sqrt{14}}$ while the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\text{proj}_{\mathbf{a}}\,\mathbf{b} = \dfrac{2}{\sqrt{14}}\dfrac{\mathbf{a}}{|\mathbf{a}|} = \frac{2}{\sqrt{14}}\cdot\frac{1}{\sqrt{14}}(\mathbf{i}+2\mathbf{j}+3\mathbf{k}) = \frac{1}{7}(\mathbf{i}+2\mathbf{j}+3\mathbf{k}) = \frac{1}{7}\mathbf{i}+\frac{2}{7}\mathbf{j}+\frac{3}{7}\mathbf{k}$.

12.4

3. $$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & -4 \\ -1 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 2 & -4 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & -4 \\ -1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix}\mathbf{k}$$
$$= [2-(-12)]\,\mathbf{i} - (0-4)\,\mathbf{j} + [0-(-2)]\,\mathbf{k} = 14\,\mathbf{i}+4\,\mathbf{j}+2\,\mathbf{k}$$

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = (14\,\mathbf{i}+4\,\mathbf{j}+2\,\mathbf{k})\cdot(2\,\mathbf{j}-4\,\mathbf{k}) = 0+8-8 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$.

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = (14\,\mathbf{i}+4\,\mathbf{j}+2\,\mathbf{k})\cdot(-\mathbf{i}+3\,\mathbf{j}+\mathbf{k}) = -14+12+2 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

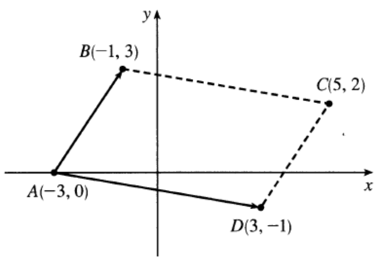
11. $(\mathbf{j}-\mathbf{i})\times(\mathbf{k}-\mathbf{i}) = (\mathbf{j}-\mathbf{i})\times\mathbf{k} + (\mathbf{j}-\mathbf{i})\times(-\mathbf{i})$ by Property 3 of Theorem 11

$= \mathbf{j}\times\mathbf{k} + (-\mathbf{i})\times\mathbf{k} + \mathbf{j}\times(-\mathbf{i}) + (-\mathbf{i})\times(-\mathbf{i})$ by Property 4 of Theorem 11

$= (\mathbf{j}\times\mathbf{k}) + (-1)(\mathbf{i}\times\mathbf{k}) + (-1)(\mathbf{j}\times\mathbf{i}) + (-1)^2(\mathbf{i}\times\mathbf{i})$ by Property 2 of Theorem 11

$= \mathbf{i} + (-1)(-\mathbf{j}) + (-1)(-\mathbf{k}) + (1)\,\mathbf{0} = \mathbf{i}+\mathbf{j}+\mathbf{k}$ by Example 2 and the discussion preceding Theorem 11

27. By plotting the vertices, we can see that the parallelogram is determined by the vectors $\overrightarrow{AB} = \langle 2,3\rangle$ and $\overrightarrow{AD} = \langle 6,-1\rangle$. We know that the area of the parallelogram determined by two vectors is equal to the length of the cross product of these vectors. In order to compute the cross product, we consider the vector $\overrightarrow{AB}$ as the three-dimensional vector $\langle 2,3,0\rangle$ (and similarly for $\overrightarrow{AD}$), and then the area of parallelogram $ABCD$ is



$$\left|\overrightarrow{AB}\times\overrightarrow{AD}\right| = \left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ 6 & -1 & 0 \end{matrix}\right\| = |(0-0)\,\mathbf{i} - (0-0)\,\mathbf{j} + (-2-18)\,\mathbf{k}| = |-20\,\mathbf{k}| = 20$$

30. (a) $\overrightarrow{PQ} = \langle 1, 2, 1 \rangle$ and $\overrightarrow{PR} = \langle 5, 0, -2 \rangle$, so a vector orthogonal to the plane through P, Q, and R is

$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle (2)(-2) - (1)(0), (1)(5) - (1)(-2), (1)(0) - (2)(5) \rangle = \langle -4, 7, -10 \rangle$ [or any scalar multiple thereof].

(b) The area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is $\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = |\langle -4, 7, -10 \rangle| = \sqrt{16 + 49 + 100} = \sqrt{165}$,

so the area of triangle PQR is $\frac{1}{2}\sqrt{165}$.

33. We know that the volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the magnitude of their scalar triple product, which

$$\text{is } \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 6 & 3 & -1 \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix} = 6\begin{vmatrix} 1 & 2 \\ -2 & 5 \end{vmatrix} - 3\begin{vmatrix} 0 & 2 \\ 4 & 5 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 1 \\ 4 & -2 \end{vmatrix} = 6(5+4) - 3(0-8) - (0-4) = 82.$$

Thus the volume of the parallelepiped is 82 cubic units.

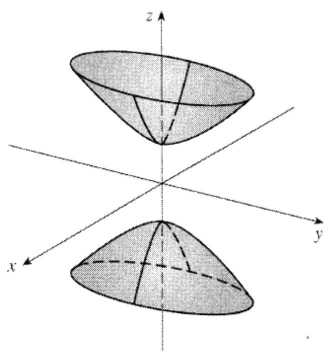
36. $\mathbf{a} = \overrightarrow{PQ} = \langle -4, 2, 4 \rangle$, $\mathbf{b} = \overrightarrow{PR} = \langle 2, 1, -2 \rangle$ and $\mathbf{c} = \overrightarrow{PS} = \langle -3, 4, 1 \rangle$.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} -4 & 2 & 4 \\ 2 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix} = -4\begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & -2 \\ -3 & 1 \end{vmatrix} + 4\begin{vmatrix} 2 & 1 \\ -3 & 4 \end{vmatrix} = -36 + 8 + 44 = 16,$$ so the volume of the

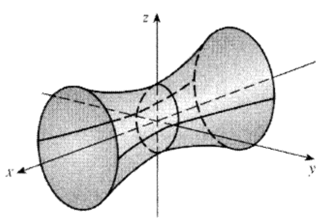
parallelepiped is 16 cubic units.

12.6

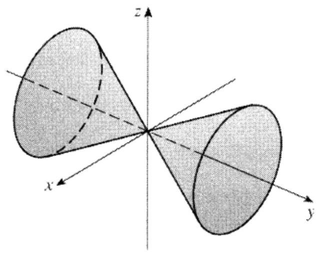
14. $z^2 - 4x^2 - y^2 = 4$. The traces in $x = k$ are the hyperbolas $z^2 - y^2 = 4 + 4k^2$, and the traces in $y = k$ are the hyperbolas $z^2 - 4x^2 = 4 + k^2$. The traces in $z = k$ are $4x^2 + y^2 = k^2 - 4$, a family of ellipses for $|k| > 2$. (The traces are a single point for $|k| = 2$ and are empty for $|k| < 2$.) The surface is a hyperboloid of two sheets with axis the z-axis.



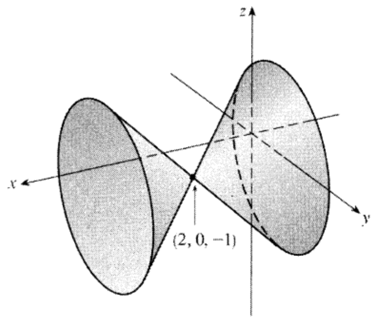
15. $9y^2 + 4z^2 = x^2 + 36$. The traces in $x = k$ are $9y^2 + 4z^2 = k^2 + 36$, a family of ellipses. The traces in $y = k$ are $4z^2 - x^2 = 9(4 - k^2)$, a family of hyperbolas for $|k| \neq 2$ and two intersecting lines when $|k| = 2$. (Note that the hyperbolas are oriented differently for $|k| < 2$ than for $|k| > 2$.) The traces in $z = k$ are $9y^2 - x^2 = 4(9 - k^2)$, a family of hyperbolas when $|k| \neq 3$ (oriented differently for $|k| < 3$ than for $|k| > 3$) and two intersecting lines when $|k| = 3$. We recognize the graph as a hyperboloid of one sheet with axis the x-axis.



18. $3x^2 - y^2 + 3z^2 = 0$. The traces in $x = k$ are $y^2 - 3z^2 = 3k^2$, a family of hyperbolas for $k \neq 0$ and two intersecting lines if $k = 0$. Traces in $y = k$ are the circles $3x^2 + 3z^2 = k^2 \quad \Leftrightarrow \quad x^2 + z^2 = \frac{1}{3}k^2$. The traces in $z = k$ are $y^2 - 3x^2 = 3k^2$, hyperbolas for $k \neq 0$ and two intersecting lines if $k = 0$. We recognize the surface as a circular cone with axis the y-axis and vertex the origin.



36. Completing squares in x and z gives $(x^2 - 4x + 4) - y^2 - (z^2 + 2z + 1) + 3 = 0 + 4 - 1 \quad \Leftrightarrow$ $(x-2)^2 - y^2 - (z+1)^2 = 0$ or $(x-2)^2 = y^2 + (z+1)^2$, a circular cone with vertex $(2, 0, -1)$ and axis the horizontal line $y = 0$, $z = -1$.



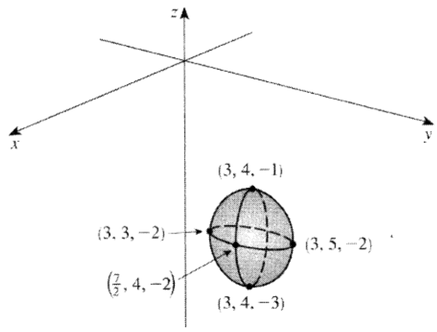
38. Completing squares in all three variables gives

$4(x^2 - 6x + 9) + (y^2 - 8y + 16) + (z^2 + 4z + 4) = -55 + 36 + 16 + 4 \quad \Leftrightarrow$

$4(x-3)^2 + (y-4)^2 + (z+2)^2 = 1$ or

$\dfrac{(x-3)^2}{1/4} + (y-4)^2 + (z+2)^2 = 1$, an ellipsoid with

center $(3, 4, -2)$.



13.2

10. $\mathbf{r}(t) = \langle e^{-t}, t - t^3, \ln t \rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \langle -e^{-t}, 1 - 3t^2, 1/t \rangle$

11. $\mathbf{r}(t) = t^2\,\mathbf{i} + \cos(t^2)\,\mathbf{j} + \sin^2 t\,\mathbf{k} \quad \Rightarrow$

$\mathbf{r}'(t) = 2t\,\mathbf{i} + [-\sin(t^2) \cdot 2t]\,\mathbf{j} + (2\sin t \cdot \cos t)\,\mathbf{k} = 2t\,\mathbf{i} - 2t\sin(t^2)\,\mathbf{j} + 2\sin t\cos t\,\mathbf{k}$

18. $\mathbf{r}(t) = \langle \tan^{-1} t, 2e^{2t}, 8te^t \rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \langle 1/(1+t^2), 4e^{2t}, 8te^t + 8e^t \rangle \quad \Rightarrow \quad \mathbf{r}'(0) = \langle 1, 4, 8 \rangle.$

So $|\mathbf{r}'(0)| = \sqrt{1^2 + 4^2 + 8^2} = \sqrt{81} = 9$ and $\mathbf{T}(0) = \dfrac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{1}{9}\langle 1, 4, 8 \rangle = \langle \frac{1}{9}, \frac{4}{9}, \frac{8}{9} \rangle$.

22. $\mathbf{r}(t) = \langle e^{2t}, e^{-2t}, te^{2t} \rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \langle 2e^{2t}, -2e^{-2t}, (2t+1)e^{2t} \rangle \quad \Rightarrow \quad \mathbf{r}'(0) = \langle 2e^0, -2e^0, (0+1)e^0 \rangle = \langle 2, -2, 1 \rangle$

and $|\mathbf{r}'(0)| = \sqrt{2^2 + (-2)^2 + 1^2} = 3$. Then $\mathbf{T}(0) = \dfrac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{1}{3}\langle 2, -2, 1 \rangle = \langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \rangle$.

$\mathbf{r}''(t) = \langle 4e^{2t}, 4e^{-2t}, (4t+4)e^{2t} \rangle \quad \Rightarrow \quad \mathbf{r}''(0) = \langle 4e^0, 4e^0, (0+4)e^0 \rangle = \langle 4, 4, 4 \rangle.$

$$\begin{aligned}
\mathbf{r}'(t) \cdot \mathbf{r}''(t) &= \langle 2e^{2t}, -2e^{-2t}, (2t+1)e^{2t} \rangle \cdot \langle 4e^{2t}, 4e^{-2t}, (4t+4)e^{2t} \rangle \\
&= (2e^{2t})(4e^{2t}) + (-2e^{-2t})(4e^{-2t}) + ((2t+1)e^{2t})((4t+4)e^{2t}) \\
&= 8e^{4t} - 8e^{-4t} + (8t^2 + 12t + 4)e^{4t} = (8t^2 + 12t + 12)e^{4t} - 8e^{-4t}
\end{aligned}$$

35.
$$\begin{aligned}
\int_0^1 (6t^2\,\mathbf{i} + t\,\mathbf{j} - 8t^3\,\mathbf{k}) &= \left(\int_0^1 6t^2\,dt\right)\mathbf{i} + \left(\int_0^1 t\,dt\right)\mathbf{j} - \left(\int_0^1 8t^3\,dt\right)\mathbf{k} \\
&= \left[2t^3\right]_0^1\,\mathbf{i} + \left[\frac{t^2}{2}\right]_0^1\,\mathbf{j} - \left[2t^4\right]_0^1\,\mathbf{k} \\
&= (2-0)\,\mathbf{i} + \left(\tfrac{1}{2} - 0\right)\mathbf{j} - (2-0)\mathbf{k} = 2\,\mathbf{i} + \tfrac{1}{2}\,\mathbf{j} - 2\,\mathbf{k}
\end{aligned}$$

36.
$$\begin{aligned}
\int_1^4 \left(2t^{3/2}\,\mathbf{i} + (t+1)\sqrt{t}\,\mathbf{k}\right)dt &= \left(\int_1^4 2t^{3/2}\,dt\right)\mathbf{i} + \left[\int_1^4 (t^{3/2} + t^{1/2})\,dt\right]\mathbf{k} \\
&= \left[\tfrac{4}{5}t^{5/2}\right]_1^4\,\mathbf{i} + \left[\tfrac{2}{5}t^{5/2} + \tfrac{2}{3}t^{3/2}\right]_1^4\,\mathbf{k} \\
&= \tfrac{4}{5}(4^{5/2} - 1)\,\mathbf{i} + \left(\tfrac{2}{5}(4)^{5/2} + \tfrac{2}{3}(4)^{3/2} - \tfrac{2}{5} - \tfrac{2}{3}\right)\mathbf{k} \\
&= \tfrac{4}{5}(31)\,\mathbf{i} + \left(\tfrac{2}{5}(32) + \tfrac{2}{3}(8) - \tfrac{2}{5} - \tfrac{2}{3}\right)\mathbf{k} = \tfrac{124}{5}\,\mathbf{i} + \tfrac{256}{15}\,\mathbf{k}
\end{aligned}$$

48.
$$\begin{aligned}
\frac{d}{dt}\left[\mathbf{u}(t) \times \mathbf{v}(t)\right] &= \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t) \qquad \text{[by Formula 5 of Theorem 3]} \\
&= \langle \cos t, -\sin t, 1 \rangle \times \langle t, \cos t, \sin t \rangle + \langle \sin t, \cos t, t \rangle \times \langle 1, -\sin t, \cos t \rangle \\
&= \left\langle -\sin^2 t - \cos t, t - \cos t\,\sin t, \cos^2 t + t\sin t \right\rangle \\
&\qquad + \left\langle \cos^2 t + t\sin t, t - \cos t\,\sin t, -\sin^2 t - \cos t \right\rangle \\
&= \left\langle \cos^2 t - \sin^2 t - \cos t + t\sin t, 2t - 2\cos t\,\sin t, \cos^2 t - \sin^2 t - \cos t + t\sin t \right\rangle
\end{aligned}$$

13.3

3. $\mathbf{r}(t) = \sqrt{2}\,t\,\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k} \quad \Rightarrow \quad \mathbf{r}'(t) = \sqrt{2}\,\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k} \quad \Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{\left(\sqrt{2}\right)^2 + (e^t)^2 + (-e^{-t})^2} = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$ [since $e^t + e^{-t} > 0$].

Then $L = \int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 (e^t + e^{-t})\,dt = \left[e^t - e^{-t}\right]_0^1 = e - e^{-1}$.

17. (a) $\mathbf{r}(t) = \langle t, 3\cos t, 3\sin t\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle 1, -3\sin t, 3\cos t\rangle \;\Rightarrow\; |\mathbf{r}'(t)| = \sqrt{1 + 9\sin^2 t + 9\cos^2 t} = \sqrt{10}$.

Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{10}}\langle 1, -3\sin t, 3\cos t\rangle$ or $\left\langle \frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}\sin t, \frac{3}{\sqrt{10}}\cos t\right\rangle$.

$\mathbf{T}'(t) = \frac{1}{\sqrt{10}}\langle 0, -3\cos t, -3\sin t\rangle \;\Rightarrow\; |\mathbf{T}'(t)| = \frac{1}{\sqrt{10}}\sqrt{0 + 9\cos^2 t + 9\sin^2 t} = \frac{3}{\sqrt{10}}$. Thus

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{1/\sqrt{10}}{3/\sqrt{10}}\langle 0, -3\cos t, -3\sin t\rangle = \langle 0, -\cos t, -\sin t\rangle.$$

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{3/\sqrt{10}}{\sqrt{10}} = \dfrac{3}{10}$

18. (a) $\mathbf{r}(t) = \langle t^2, \sin t - t\cos t, \cos t + t\sin t\rangle \;\Rightarrow$

$\mathbf{r}'(t) = \langle 2t, \cos t + t\sin t - \cos t,\; -\sin t + t\cos t + \sin t\rangle = \langle 2t, t\sin t, t\cos t\rangle \;\Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{4t^2 + t^2\sin^2 t + t^2\cos^2 t} = \sqrt{4t^2 + t^2(\cos^2 t + \sin^2 t)} = \sqrt{5t^2} = \sqrt{5}\,t$ [since $t > 0$]. Then

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{5}\,t}\langle 2t, t\sin t, t\cos t\rangle = \frac{1}{\sqrt{5}}\langle 2, \sin t, \cos t\rangle$. $\mathbf{T}'(t) = \frac{1}{\sqrt{5}}\langle 0, \cos t, -\sin t\rangle \;\Rightarrow$

$|\mathbf{T}'(t)| = \frac{1}{\sqrt{5}}\sqrt{0 + \cos^2 t + \sin^2 t} = \frac{1}{\sqrt{5}}$. Thus $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{1/\sqrt{5}}{1/\sqrt{5}}\langle 0, \cos t, -\sin t\rangle = \langle 0, \cos t, -\sin t\rangle$.

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{1/\sqrt{5}}{\sqrt{5}\,t} = \dfrac{1}{5t}$

19. (a) $\mathbf{r}(t) = \langle \sqrt{2}\,t, e^t, e^{-t}\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle \sqrt{2}, e^t, -e^{-t}\rangle \;\Rightarrow\; |\mathbf{r}'(t)| = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$

Then

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{e^t + e^{-t}}\langle \sqrt{2}, e^t, -e^{-t}\rangle = \dfrac{1}{e^{2t}+1}\langle \sqrt{2}\,e^t, e^{2t}, -1\rangle$ $\left[\text{after multiplying by } \dfrac{e^t}{e^t}\right]$ and

$$\begin{aligned}\mathbf{T}'(t) &= \frac{1}{e^{2t}+1}\langle \sqrt{2}\,e^t, 2e^{2t}, 0\rangle - \frac{2e^{2t}}{(e^{2t}+1)^2}\langle \sqrt{2}\,e^t, e^{2t}, -1\rangle \\ &= \frac{1}{(e^{2t}+1)^2}\left[(e^{2t}+1)\langle \sqrt{2}\,e^t, 2e^{2t}, 0\rangle - 2e^{2t}\langle \sqrt{2}\,e^t, e^{2t}, -1\rangle\right] = \frac{1}{(e^{2t}+1)^2}\langle \sqrt{2}\,e^t\left(1 - e^{2t}\right), 2e^{2t}, 2e^{2t}\rangle\end{aligned}$$

Then

$$\begin{aligned}|\mathbf{T}'(t)| &= \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}(1 - 2e^{2t} + e^{4t}) + 4e^{4t} + 4e^{4t}} = \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}(1 + 2e^{2t} + e^{4t})} \\ &= \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}\,(1 + e^{2t})^2} = \frac{\sqrt{2}\,e^t(1 + e^{2t})}{(e^{2t}+1)^2} = \frac{\sqrt{2}\,e^t}{e^{2t}+1}\end{aligned}$$

Therefore

$$\begin{aligned}\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} &= \frac{e^{2t}+1}{\sqrt{2}\,e^t}\frac{1}{(e^{2t}+1)^2}\langle \sqrt{2}\,e^t(1 - e^{2t}), 2e^{2t}, 2e^{2t}\rangle \\ &= \frac{1}{\sqrt{2}\,e^t(e^{2t}+1)}\langle \sqrt{2}\,e^t(1 - e^{2t}), 2e^{2t}, 2e^{2t}\rangle = \frac{1}{e^{2t}+1}\langle 1 - e^{2t}, \sqrt{2}\,e^t, \sqrt{2}\,e^t\rangle\end{aligned}$$

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{\sqrt{2}\,e^t}{e^{2t}+1}\cdot\dfrac{1}{e^t + e^{-t}} = \dfrac{\sqrt{2}\,e^t}{e^{3t} + 2e^t + e^{-t}} = \dfrac{\sqrt{2}\,e^{2t}}{e^{4t} + 2e^{2t} + 1} = \dfrac{\sqrt{2}\,e^{2t}}{(e^{2t}+1)^2}$

20. (a) $\mathbf{r}(t) = \langle t, \frac{1}{2}t^2, t^2 \rangle \Rightarrow \mathbf{r}'(t) = \langle 1, t, 2t \rangle \Rightarrow |\mathbf{r}'(t)| = \sqrt{1+t^2+4t^2} = \sqrt{1+5t^2}$. Then

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{1+5t^2}} \langle 1, t, 2t \rangle.$$

$$\mathbf{T}'(t) = \frac{-5t}{(1+5t^2)^{3/2}} \langle 1, t, 2t \rangle + \frac{1}{\sqrt{1+5t^2}} \langle 0, 1, 2 \rangle \quad \text{[by Formula 3 of Theorem 13.2.3]}$$

$$= \frac{1}{(1+5t^2)^{3/2}} \left(\langle -5t, -5t^2, -10t^2 \rangle + \langle 0, 1+5t^2, 2+10t^2 \rangle \right) = \frac{1}{(1+5t^2)^{3/2}} \langle -5t, 1, 2 \rangle$$

$$|\mathbf{T}'(t)| = \frac{1}{(1+5t^2)^{3/2}} \sqrt{25t^2+1+4} = \frac{1}{(1+5t^2)^{3/2}} \sqrt{25t^2+5} = \frac{\sqrt{5}\sqrt{5t^2+1}}{(1+5t^2)^{3/2}} = \frac{\sqrt{5}}{1+5t^2}$$

Thus $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{1+5t^2}{\sqrt{5}} \cdot \dfrac{1}{(1+5t^2)^{3/2}} \langle -5t, 1, 2 \rangle = \dfrac{1}{\sqrt{5+25t^2}} \langle -5t, 1, 2 \rangle.$

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{\sqrt{5}/(1+5t^2)}{\sqrt{1+5t^2}} = \dfrac{\sqrt{5}}{(1+5t^2)^{3/2}}$

23. $\mathbf{r}(t) = \sqrt{6}\,t^2\,\mathbf{i} + 2t\,\mathbf{j} + 2t^3\,\mathbf{k} \Rightarrow \mathbf{r}'(t) = 2\sqrt{6}\,t\,\mathbf{i} + 2\,\mathbf{j} + 6t^2\,\mathbf{k}$, $\mathbf{r}''(t) = 2\sqrt{6}\,\mathbf{i} + 12t\,\mathbf{k}$,

$|\mathbf{r}'(t)| = \sqrt{24t^2+4+36t^4} = \sqrt{4(9t^4+6t^2+1)} = \sqrt{4(3t^2+1)^2} = 2(3t^2+1)$,

$\mathbf{r}'(t) \times \mathbf{r}''(t) = 24t\,\mathbf{i} - 12\sqrt{6}\,t^2\,\mathbf{j} - 4\sqrt{6}\,\mathbf{k}$,

$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{576t^2+864t^4+96} = \sqrt{96(9t^4+6t^2+1)} = \sqrt{96(3t^2+1)^2} = 4\sqrt{6}\,(3t^2+1)$.

Then $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{4\sqrt{6}\,(3t^2+1)}{8(3t^2+1)^3} = \dfrac{\sqrt{6}}{2(3t^2+1)^2}$.

24. $\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t, t \rangle \Rightarrow \mathbf{r}'(t) = \langle e^t \cos t - e^t \sin t, e^t \cos t + e^t \sin t, 1 \rangle$. The point $(1,0,0)$ corresponds to $t = 0$, and $\mathbf{r}'(0) = \langle 1,1,1 \rangle \Rightarrow |\mathbf{r}'(0)| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$.

$\mathbf{r}''(t) = \langle e^t \cos t - e^t \sin t - e^t \cos t - e^t \sin t, e^t \cos t - e^t \sin t + e^t \cos t + e^t \sin t, 0 \rangle = \langle -2e^t \sin t, 2e^t \cos t, 0 \rangle \Rightarrow$

$\mathbf{r}''(0) = \langle 0, 2, 0 \rangle$. $\mathbf{r}'(0) \times \mathbf{r}''(0) = \langle -2, 0, 2 \rangle$. $|\mathbf{r}'(0) \times \mathbf{r}''(0)| = \sqrt{(-2)^2+0^2+2^2} = \sqrt{8} = 2\sqrt{2}$.

Then $\kappa(0) = \dfrac{|\mathbf{r}'(0) \times \mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3} = \dfrac{2\sqrt{2}}{(\sqrt{3})^3} = \dfrac{2\sqrt{2}}{3\sqrt{3}}$ or $\dfrac{2\sqrt{6}}{9}$.

27. $f(x) = x^3$, $f'(x) = 3x^2$, $f''(x) = 6x$, $\kappa(x) = \dfrac{|f''(x)|}{[1+(f'(x))^2]^{3/2}} = \dfrac{|6x|}{[1+(3x^2)^2]^{3/2}} = \dfrac{6\,|x|}{(1+9x^4)^{3/2}}$

28. $f(x) = \cos x$, $f'(x) = -\sin x$, $f''(x) = -\cos x$, $\kappa(x) = \dfrac{|f''(x)|}{[1+(f'(x))^2]^{3/2}} = \dfrac{|-\cos x|}{[1+(-\sin x)^2]^{3/2}} = \dfrac{|\cos x|}{(1+\sin^2 x)^{3/2}}$

14.2

6. $f(x,y) = \dfrac{x^2y+xy^2}{x^2-y^2}$ is a rational function and hence continuous on its domain.

$(2,-1)$ is in the domain of f, so f is continuous there and $\displaystyle\lim_{(x,y)\to(2,-1)} f(x,y) = f(2,-1) = \frac{(2)^2(-1)+(2)(-1)^2}{(2)^2-(-1)^2} = -\frac{2}{3}$.

7. $x - y$ is a polynomial and therefore continuous. Since $\sin t$ is a continuous function, the composition $\sin(x-y)$ is also continuous. The function y is a polynomial, and hence continuous, and the product of continuous functions is continuous, so $f(x,y) = y\sin(x-y)$ is a continuous function. Then $\lim\limits_{(x,y)\to(\pi,\pi/2)} f(x,y) = f\left(\pi, \frac{\pi}{2}\right) = \frac{\pi}{2}\sin\left(\pi - \frac{\pi}{2}\right) = \frac{\pi}{2}\sin\frac{\pi}{2} = \frac{\pi}{2}$.

8. $2x - y$ is a polynomial and therefore continuous. Since $\sqrt{t}$ is continuous for $t \geq 0$, the composition $\sqrt{2x-y}$ is continuous where $2x - y \geq 0$. The function e^u is continuous everywhere, so the composition $f(x,y) = e^{\sqrt{2x-y}}$ is a continuous function for $2x - y \geq 0$. If $x = 3$ and $y = 2$ then $2x - y \geq 0$, so $\lim\limits_{(x,y)\to(3,2)} f(x,y) = f(3,2) = e^{\sqrt{2(3)-2}} = e^2$.

9. $f(x,y) = (x^4 - 4y^2)/(x^2 + 2y^2)$. First approach $(0,0)$ along the x-axis. Then $f(x,0) = x^4/x^2 = x^2$ for $x \neq 0$, so $f(x,y) \to 0$. Now approach $(0,0)$ along the y-axis. For $y \neq 0$, $f(0,y) = -4y^2/2y^2 = -2$, so $f(x,y) \to -2$. Since f has two different limits along two different lines, the limit does not exist.

12. $f(x,y) = \dfrac{x^4 - y^4}{x^2 + y^2} = \dfrac{(x^2+y^2)(x^2-y^2)}{x^2+y^2} = x^2 - y^2$ for $(x,y) \neq (0,0)$. Thus the limit as $(x,y) \to (0,0)$ is 0.

18. We can use the Squeeze Theorem to show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2\sin^2 y}{x^2 + 2y^2} = 0$:

$0 \leq \dfrac{x^2 \sin^2 y}{x^2 + 2y^2} \leq \sin^2 y$ since $\dfrac{x^2}{x^2+2y^2} \leq 1$, and $\sin^2 y \to 0$ as $(x,y) \to (0,0)$, so $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2\sin^2 y}{x^2+2y^2} = 0$.

31. $F(x,y) = \dfrac{1 + x^2 + y^2}{1 - x^2 - y^2}$ is a rational function and thus is continuous on its domain

$\{(x,y) \mid 1 - x^2 - y^2 \neq 0\} = \{(x,y) \mid x^2 + y^2 \neq 1\}$.

38. $f(x,y) = \begin{cases} \dfrac{xy}{x^2+xy+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ The first piece of f is a rational function defined everywhere except at the origin, so f is continuous on $\mathbb{R}^2$ except possibly at the origin. $f(x,0) = 0/x^2 = 0$ for $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. But $f(x,x) = x^2/(3x^2) = \frac{1}{3}$ for $x \neq 0$, so $f(x,y) \to \frac{1}{3}$ as $(x,y) \to (0,0)$ along the line $y = x$. Thus $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ doesn't exist, so f is not continuous at $(0,0)$ and the largest set on which f is continuous is $\{(x,y) \mid (x,y) \neq (0,0)\}$.

14.3

21. $f(x,y) = x/y = xy^{-1} \Rightarrow f_x(x,y) = y^{-1} = 1/y,\ f_y(x,y) = -xy^{-2} = -x/y^2$

22. $f(x,t) = \sqrt{x}\ln t \Rightarrow f_x(x,t) = \frac{1}{2}x^{-1/2}\ln t = (\ln t)/(2\sqrt{x}),\ f_t(x,t) = \sqrt{x} \cdot \dfrac{1}{t} = \sqrt{x}/t$

23. $f(x,t) = e^{-t}\cos \pi x \Rightarrow f_x(x,t) = e^{-t}(-\sin \pi x)(\pi) = -\pi e^{-t}\sin \pi x,\ f_t(x,t) = e^{-t}(-1)\cos \pi x = -e^{-t}\cos \pi x$

24. $z = \tan xy \Rightarrow \dfrac{\partial z}{\partial x} = (\sec^2 xy)(y) = y\sec^2 xy,\ \dfrac{\partial z}{\partial y} = (\sec^2 xy)(x) = x\sec^2 xy$

28. $f(x,y) = x^y \;\Rightarrow\; f_x(x,y) = yx^{y-1},\; f_y(x,y) = x^y \ln x$

29. $F(x,y) = \int_y^x \cos(e^t)\,dt \;\Rightarrow\; F_x(x,y) = \frac{\partial}{\partial x}\int_y^x \cos(e^t)\,dt = \cos(e^x)$ by the Fundamental Theorem of Calculus, Part 1;

$$F_y(x,y) = \frac{\partial}{\partial y}\int_y^x \cos(e^t)\,dt = \frac{\partial}{\partial y}\left[-\int_x^y \cos(e^t)\,dt\right] = -\frac{\partial}{\partial y}\int_x^y \cos(e^t)\,dt = -\cos(e^y).$$

42. $f(x,y) = y\sin^{-1}(xy) \;\Rightarrow\; f_y(x,y) = y\cdot\frac{1}{\sqrt{1-(xy)^2}}(x) + \sin^{-1}(xy)\cdot 1 = \frac{xy}{\sqrt{1-x^2y^2}} + \sin^{-1}(xy),$

so $f_y\left(1,\frac{1}{2}\right) = \frac{1\cdot\frac{1}{2}}{\sqrt{1-1^2\left(\frac{1}{2}\right)^2}} + \sin^{-1}\left(1\cdot\frac{1}{2}\right) = \frac{\frac{1}{2}}{\sqrt{\frac{3}{4}}} + \sin^{-1}\frac{1}{2} = \frac{1}{\sqrt{3}} + \frac{\pi}{6}.$

47. $x^2+2y^2+3z^2 = 1 \;\Rightarrow\; \frac{\partial}{\partial x}(x^2+2y^2+3z^2) = \frac{\partial}{\partial x}(1) \;\Rightarrow\; 2x + 0 + 6z\frac{\partial z}{\partial x} = 0 \;\Rightarrow\; 6z\frac{\partial z}{\partial x} = -2x \;\Rightarrow$

$\frac{\partial z}{\partial x} = \frac{-2x}{6z} = -\frac{x}{3z}$, and $\frac{\partial}{\partial y}(x^2+2y^2+3z^2) = \frac{\partial}{\partial y}(1) \;\Rightarrow\; 0 + 4y + 6z\frac{\partial z}{\partial y} = 0 \;\Rightarrow\; 6z\frac{\partial z}{\partial y} = -4y \;\Rightarrow$

$\frac{\partial z}{\partial y} = \frac{-4y}{6z} = -\frac{2y}{3z}.$

48. $x^2 - y^2 + z^2 - 2z = 4 \;\Rightarrow\; \frac{\partial}{\partial x}(x^2 - y^2 + z^2 - 2z) = \frac{\partial}{\partial x}(4) \;\Rightarrow\; 2x - 0 + 2z\frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial x} = 0 \;\Rightarrow$

$(2z-2)\frac{\partial z}{\partial x} = -2x \;\Rightarrow\; \frac{\partial z}{\partial x} = \frac{-2x}{2z-2} = \frac{x}{1-z}$, and $\frac{\partial}{\partial y}(x^2 - y^2 + z^2 - 2z) = \frac{\partial}{\partial y}(4) \;\Rightarrow$

$0 - 2y + 2z\frac{\partial z}{\partial y} - 2\frac{\partial z}{\partial y} = 0 \;\Rightarrow\; (2z-2)\frac{\partial z}{\partial y} = 2y \;\Rightarrow\; \frac{\partial z}{\partial y} = \frac{2y}{2z-2} = \frac{y}{z-1}.$

57. $v = \sin(s^2 - t^2) \;\Rightarrow\; v_s = \cos(s^2-t^2)\cdot 2s = 2s\cos(s^2-t^2),\; v_t = \cos(s^2-t^2)\cdot(-2t) = -2t\cos(s^2-t^2)$. Then

$v_{ss} = 2s\left[-\sin(s^2-t^2)\cdot 2s\right] + \cos(s^2-t^2)\cdot 2 = 2\cos(s^2-t^2) - 4s^2\sin(s^2-t^2),$

$v_{st} = 2s\left[-\sin(s^2-t^2)\cdot(-2t)\right] = 4st\sin(s^2-t^2),\quad v_{ts} = -2t\left[-\sin(s^2-t^2)\cdot 2s\right] = 4st\sin(s^2-t^2),$

$v_{tt} = -2t\cdot\left[-\sin(s^2-t^2)\cdot(-2t)\right] + \cos(s^2-t^2)\cdot(-2) = -2\cos(s^2-t^2) - 4t^2\sin(s^2-t^2).$

58. $w = \sqrt{1+uv^2} \quad \Rightarrow \quad w_u = \frac{1}{2}(1+uv^2)^{-1/2} \cdot v^2 = \dfrac{v^2}{2\sqrt{1+uv^2}}, \quad w_v = \frac{1}{2}(1+uv^2)^{-1/2} \cdot 2uv = \dfrac{uv}{\sqrt{1+uv^2}}.$

Then $w_{uu} = \frac{1}{2}v^2\left(-\frac{1}{2}\right)(1+uv^2)^{-3/2}(v^2) = -\dfrac{v^4}{4(1+uv^2)^{3/2}},$

$$w_{uv} = \frac{2\sqrt{1+uv^2} \cdot 2v - v^2 \cdot 2\left(\frac{1}{2}\right)(1+uv^2)^{-1/2}(2uv)}{\left(2\sqrt{1+uv^2}\right)^2} = \frac{4v\sqrt{1+uv^2} - 2uv^3/\sqrt{1+uv^2}}{4(1+uv^2)}$$

$$= \frac{4v(1+uv^2) - 2uv^3}{4\,(1+uv^2)^{3/2}} = \frac{2v+uv^3}{2\,(1+uv^2)^{3/2}}$$

$$w_{vu} = \frac{\sqrt{1+uv^2} \cdot v - uv \cdot \frac{1}{2}(1+uv^2)^{-1/2}(v^2)}{\left(\sqrt{1+uv^2}\right)^2} = \frac{v\sqrt{1+uv^2} - \frac{1}{2}uv^3/\sqrt{1+uv^2}}{(1+uv^2)}$$

$$= \frac{v(1+uv^2) - \frac{1}{2}uv^3}{(1+uv^2)^{3/2}} = \frac{2v+uv^3}{2\,(1+uv^2)^{3/2}}$$

$$w_{vv} = \frac{\sqrt{1+uv^2} \cdot u - uv \cdot \frac{1}{2}(1+uv^2)^{-1/2}(2uv)}{\left(\sqrt{1+uv^2}\right)^2} = \frac{u\sqrt{1+uv^2} - u^2v^2/\sqrt{1+uv^2}}{(1+uv^2)}$$

$$= \frac{u(1+uv^2) - u^2v^2}{(1+uv^2)^{3/2}} = \frac{u}{(1+uv^2)^{3/2}}$$

68. $V = \ln(r+s^2+t^3) \quad \Rightarrow \quad \dfrac{\partial V}{\partial t} = \dfrac{3t^2}{r+s^2+t^3} = 3t^2(r+s^2+t^3)^{-1},$

$$\frac{\partial^2 V}{\partial s\,\partial t} = 3t^2(-1)(r+s^2+t^3)^{-2}(2s) = -6st^2(r+s^2+t^3)^{-2},$$

$$\frac{\partial^3 V}{\partial r\,\partial s\,\partial t} = -6st^2(-2)(r+s^2+t^3)^{-3}(1) = 12st^2(r+s^2+t^3)^{-3} = \frac{12st^2}{(r+s^2+t^3)^3}.$$

69. $u = e^{r\theta}\sin\theta \quad \Rightarrow \quad \dfrac{\partial u}{\partial \theta} = e^{r\theta}\cos\theta + \sin\theta \cdot e^{r\theta}\,(r) = e^{r\theta}\,(\cos\theta + r\sin\theta),$

$$\frac{\partial^2 u}{\partial r\,\partial\theta} = e^{r\theta}\,(\sin\theta) + (\cos\theta + r\sin\theta)\,e^{r\theta}\,(\theta) = e^{r\theta}\,(\sin\theta + \theta\cos\theta + r\theta\sin\theta),$$

$$\frac{\partial^3 u}{\partial r^2\,\partial\theta} = e^{r\theta}\,(\theta\sin\theta) + (\sin\theta + \theta\cos\theta + r\theta\sin\theta) \cdot e^{r\theta}\,(\theta) = \theta e^{r\theta}\,(2\sin\theta + \theta\cos\theta + r\theta\sin\theta).$$

14.4

3. $z = f(x,y) = e^{x-y} \quad \Rightarrow \quad f_x(x,y) = e^{x-y}(1) = e^{x-y}, \quad f_y(x,y) = e^{x-y}(-1) = -e^{x-y},$ so $f_x(2,2) = 1$ and $f_y(2,2) = -1$. Thus an equation of the tangent plane is $z - 1 = f_x(2,2)(x-2) + f_y(2,2)(y-2) \quad \Rightarrow$ $z - 1 = 1(x-2) + (-1)(y-2)$ or $z = x - y + 1$.

4. $z = f(x,y) = x/y^2 = xy^{-2} \quad \Rightarrow \quad f_x(x,y) = 1/y^2, \quad f_y(x,y) = -2xy^{-3} = -2x/y^3,$ so $f_x(-4,2) = \frac{1}{4}$ and $f_y(-4,2) = 1$. Thus an equation of the tangent plane is $z - (-1) = f_x(-4,2)\,[x-(-4)] + f_y(-4,2)(y-2) \quad \Rightarrow$ $z + 1 = \frac{1}{4}(x+4) + 1(y-2)$ or $z = \frac{1}{4}x + y - 2$.

12. $f(x,y)=\sqrt{xy}=(xy)^{1/2}$. The partial derivatives are $f_x(x,y)=\frac{1}{2}(xy)^{-1/2}(y)=y/\left(2\sqrt{xy}\right)$ and $f_y(x,y)=\frac{1}{2}(xy)^{-1/2}(x)=x/\left(2\sqrt{xy}\right)$, so $f_x(1,4)=4/\left(2\sqrt{4}\right)=1$ and $f_y(1,4)=1/\left(2\sqrt{4}\right)=\frac{1}{4}$. Both f_x and f_y are continuous functions for $xy>0$, so f is differentiable at $(1,4)$ by Theorem 8. The linearization of f at $(1,4)$ is $L(x,y)=f(1,4)+f_x(1,4)(x-1)+f_y(1,4)(y-4)=2+1(x-1)+\frac{1}{4}(y-4)=x+\frac{1}{4}y$.

13. $f(x,y)=x^2e^y$. The partial derivatives are $f_x(x,y)=2xe^y$ and $f_y(x,y)=x^2e^y$, so $f_x(1,0)=2$ and $f_y(1,0)=1$. Both f_x and f_y are continuous functions, so by Theorem 8, f is differentiable at $(1,0)$. By Equation 3, the linearization of f at $(1,0)$ is given by $L(x,y)=f(1,0)+f_x(1,0)(x-1)+f_y(1,0)(y-0)=1+2(x-1)+1(y-0)=2x+y-1$.

17. Let $f(x,y)=e^x\cos(xy)$. Then $f_x(x,y)=e^x[-\sin(xy)](y)+e^x\cos(xy)=e^x[\cos(xy)-y\sin(xy)]$ and $f_y(x,y)=e^x[-\sin(xy)](x)=-xe^x\sin(xy)$. Both f_x and f_y are continuous functions, so by Theorem 8, f is differentiable at $(0,0)$. We have $f_x(0,0)=e^0(\cos 0-0)=1$, $f_y(0,0)=0$ and the linear approximation of f at $(0,0)$ is $f(x,y)\approx f(0,0)+f_x(0,0)(x-0)+f_y(0,0)(y-0)=1+1x+0y=x+1$.

18. Let $f(x,y)=\dfrac{y-1}{x+1}$. Then $f_x(x,y)=(y-1)(-1)(x+1)^{-2}=\dfrac{1-y}{(x+1)^2}$ and $f_y(x,y)=\dfrac{1}{x+1}$. Both f_x and f_y are continuous functions for $x\neq -1$, so by Theorem 8, f is differentiable at $(0,0)$. We have $f_x(0,0)=1$, $f_y(0,0)=1$ and the linear approximation of f at $(0,0)$ is $f(x,y)\approx f(0,0)+f_x(0,0)(x-0)+f_y(0,0)(y-0)=-1+1x+1y=x+y-1$.

25. $z=e^{-2x}\cos 2\pi t \quad\Rightarrow$

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial t}dt=e^{-2x}(-2)\cos 2\pi t\,dx+e^{-2x}(-\sin 2\pi t)(2\pi)\,dt=-2e^{-2x}\cos 2\pi t\,dx-2\pi e^{-2x}\sin 2\pi t\,dt$$

26. $u=\sqrt{x^2+3y^2}=(x^2+3y^2)^{1/2} \quad\Rightarrow$

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=\tfrac{1}{2}(x^2+3y^2)^{-1/2}(2x)\,dx+\tfrac{1}{2}(x^2+3y^2)^{-1/2}(6y)\,dy=\frac{x}{\sqrt{x^2+3y^2}}\,dx+\frac{3y}{\sqrt{x^2+3y^2}}\,dy$$

33. $dA=\dfrac{\partial A}{\partial x}\,dx+\dfrac{\partial A}{\partial y}\,dy=y\,dx+x\,dy$ and $|\Delta x|\le 0.1$, $|\Delta y|\le 0.1$. We use $dx=0.1$, $dy=0.1$ with $x=30$, $y=24$; then the maximum error in the area is about $dA=24(0.1)+30(0.1)=5.4\text{ cm}^2$.

34. Let V be the volume. Then $V=\pi r^2h$ and $\Delta V\approx dV=2\pi rh\,dr+\pi r^2\,dh$ is an estimate of the amount of metal. With $dr=0.05$ and $dh=0.2$ we get $dV=2\pi(2)(10)(0.05)+\pi(2)^2(0.2)=2.80\pi\approx 8.8\text{ cm}^3$.

14.5

5. $w=xe^{y/z}$, $x=t^2$, $y=1-t$, $z=1+2t \quad\Rightarrow$

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}=e^{y/z}\cdot 2t+xe^{y/z}\left(\frac{1}{z}\right)\cdot(-1)+xe^{y/z}\left(-\frac{y}{z^2}\right)\cdot 2=e^{y/z}\left(2t-\frac{x}{z}-\frac{2xy}{z^2}\right)$$

6. $z=\tan^{-1}(y/x)$, $x=e^t$, $y=1-e^{-t} \quad\Rightarrow$

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=\frac{1}{1+(y/x)^2}(-yx^{-2})\cdot e^t+\frac{1}{1+(y/x)^2}(1/x)\cdot(-e^{-t})(-1)$$

$$=-\frac{y}{x^2+y^2}\cdot e^t+\frac{1}{x+y^2/x}\cdot e^{-t}=\frac{xe^{-t}-ye^t}{x^2+y^2}$$

9. $z = \ln(3x + 2y), \quad x = s\sin t, \quad y = t\cos s \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{3}{3x+2y}(\sin t) + \frac{2}{3x+2y}(-t\sin s) = \frac{3\sin t - 2t\sin s}{3x+2y}$$

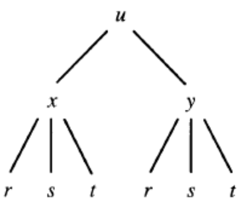
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{3}{3x+2y}(s\cos t) + \frac{2}{3x+2y}(\cos s) = \frac{3s\cos t + 2\cos s}{3x+2y}$$

10. $z = \sqrt{x}\,e^{xy}, \quad x = 1 + st, \quad y = s^2 - t^2 \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \left(\sqrt{x}\cdot e^{xy}(y) + e^{xy}\cdot \tfrac{1}{2}x^{-1/2}\right)(t) + \sqrt{x}\,e^{xy}(x)\,(2s) = \left(yt\sqrt{x} + \frac{t}{2\sqrt{x}} + 2x^{3/2}s\right)e^{xy}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \left(\sqrt{x}\cdot e^{xy}(y) + e^{xy}\cdot \tfrac{1}{2}x^{-1/2}\right)(s) + \sqrt{x}\,e^{xy}(x)\,(-2t) = \left(ys\sqrt{x} + \frac{s}{2\sqrt{x}} - 2x^{3/2}t\right)e^{xy}$$

17.



$u = f(x, y), \ x = x(r, s, t), \ y = y(r, s, t) \quad \Rightarrow$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}, \quad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s},$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

21. $z = x^2 + xy^3, \ x = uv^2 + w^3, \ y = u + ve^w \quad \Rightarrow$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (2x + y^3)(v^2) + (3xy^2)(1),$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = (2x + y^3)(2uv) + (3xy^2)(e^w),$$

$$\frac{\partial z}{\partial w} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial w} = (2x + y^3)(3w^2) + (3xy^2)(ve^w).$$

When $u = 2$, $v = 1$, and $w = 0$, we have $x = 2$, $y = 3$,

so $\dfrac{\partial z}{\partial u} = (31)(1) + (54)(1) = 85$, $\dfrac{\partial z}{\partial v} = (31)\,(4) + (54)(1) = 178$, $\dfrac{\partial z}{\partial w} = (31)(0) + (54)(1) = 54$.

22. $u = (r^2 + s^2)^{1/2}, \ r = y + x\cos t, \ s = x + y\sin t \quad \Rightarrow$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = \tfrac{1}{2}(r^2 + s^2)^{-1/2}(2r)(\cos t) + \tfrac{1}{2}(r^2 + s^2)^{-1/2}(2s)(1) = (r\cos t + s)/\sqrt{r^2 + s^2},$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} = \tfrac{1}{2}(r^2 + s^2)^{-1/2}(2r)(1) + \tfrac{1}{2}(r^2 + s^2)^{-1/2}(2s)(\sin t) = (r + s\sin t)/\sqrt{r^2 + s^2},$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial t} = \tfrac{1}{2}(r^2 + s^2)^{-1/2}(2r)(-x\sin t) + \tfrac{1}{2}(r^2 + s^2)^{-1/2}(2s)(y\cos t) = \frac{-rx\sin t + sy\cos t}{\sqrt{r^2 + s^2}}.$$

When $x = 1$, $y = 2$, and $t = 0$ we have $r = 3$ and $s = 1$, so $\dfrac{\partial u}{\partial x} = \dfrac{4}{\sqrt{10}}$, $\dfrac{\partial u}{\partial y} = \dfrac{3}{\sqrt{10}}$, and $\dfrac{\partial u}{\partial t} = \dfrac{2}{\sqrt{10}}$.

27. $y\cos x = x^2 + y^2$, so let $F(x, y) = y\cos x - x^2 - y^2 = 0$. Then by Equation 6

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-y\sin x - 2x}{\cos x - 2y} = \frac{2x + y\sin x}{\cos x - 2y}.$$

32. $x^2 - y^2 + z^2 - 2z = 4$, so let $F(x, y, z) = x^2 - y^2 + z^2 - 2z - 4 = 0$. Then by Equations 7

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{2z - 2} = \frac{x}{1 - z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-2y}{2z - 2} = \frac{y}{z - 1}.$$