A MAPPING PROPERTY OF THE BERGMAN PROJECTION ON CERTAIN PSEUDOCONVEX DOMAINS

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Abstract. We show that the Bergman kernel function, associated to pseudoconvex domains of finite type with the property that the Levi form of the boundary has at most one degenerate eigenvalue, is a standard kernel of Calderón-Zygmund type with respect to the Lebesgue measure. As an application, we show that the Bergman projection on these domains preserves some of the Lebesgue classes.

1. Introduction.

Let Ω ⊂ C^n be a bounded domain. The Bergman projection P on Ω is the orthogonal projection

\[ P : L^2(Ω) \longrightarrow H(Ω) \cap L^2(Ω) = A^2(Ω), \]

where H(Ω) denotes the set of holomorphic functions on Ω. There is a corresponding kernel function \( K_\Omega(z, w) \), the Bergman kernel function, such that

\[ Pf(z) = \int_Ω K_\Omega(z, w)f(w)dw. \]

Let a triple \((S, d, \mu)\) be a space of homogeneous type, that is, \( S \) is a set, \( d \) is a pseudometric on \( S \) and \( \mu \) is a positive measure on \( S \); more precisely, \( d : S \times S \to [0, \infty) \) satisfies

(a) \( d(x, y) = 0 \iff x = y \),
(b) \( C_1^{-1}d(y, x) \leq d(x, y) \leq C_1d(y, x) \),
(c) \( d(x, y) \leq C_2(d(x, z) + d(z, y)) \) for \( x, y, z \in S \),

for independent constants \( C_1, C_2 \); and for all \( x \in S \) and small \( \delta > 0 \), there is an independent constant \( C_3 \) such that

(i) \( \mu(P(x, \delta)) < \infty \);
(ii) \( \mu(P(x, 2\delta)) \leq C_3\mu(P(x, \delta)) \),

where

\[ P(x, \delta) = \{ y \in S : d(x, y) < \delta \}. \]

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Definition 1.1. A kernel $K : S \times S - \{x = y\} \to \mathbb{C}$ is called a standard kernel if there exist independent constants $T > 0$ and $C < \infty$ such that for all $x \neq y \in S$,

$$|K(x, y)| \leq \frac{C}{\mu(P(x, d(x, y)))}$$

and for all $x, z \in S$,

$$\int_{d(x, y) > Td(x, z)} |K(x, y) - K(z, y)|dy \leq C.$$

In all that follows, we assume that $\Omega$ is a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth defining function $r$. We also assume that all the points of $b\Omega$ are of finite type in the sense of D’Angelo [4], and the Levi form $\partial \bar{\partial} r(z)$ of $b\Omega$ has at least $(n - 2)$-positive eigenvalues at every point $z \in b\Omega$.

Theorem 1.2. Let $\Omega$ be as above. Then the Bergman kernel $K_{\Omega}(z', z)$ is a standard kernel with respect to a pseudometric $d$ and the Lebesgue measure $\mu$.

Here $d$ is a pseudometric to be determined explicitly. As an application, we prove:

Theorem 1.3. Let $\Omega$ be as above. Then the Bergman projection $P$ is bounded on $L^p(\Omega), 1 < p < \infty$.

For geometrically convex domains of finite type in $\mathbb{C}^n$, McNeal [5] showed that the Bergman kernel is a standard kernel and is bounded in $L^p(\Omega), 1 < p < \infty$. He also mentioned that the same results hold for pseudoconvex domains of finite type in $\mathbb{C}^2$ and for decoupled pseudoconvex domains of finite type in $\mathbb{C}^n$. The main technical difficulties in proving these theorems are to construct a suitable pseudometric $d$ on $\Omega$ with “doubling property” of the balls, and to estimate $|K(z', z) - K(w, z)|$ whenever $z$ satisfies $d(z', z) > Td(z', w)$ for some large $T$. The “doubling property” in our case is proved in Section 2 (Proposition 2.5). To estimate $|K(z', z) - K(w, z)|$, we will use the estimates of the Bergman kernel and its derivatives (cf. [1], [2]) of the domain we are considering.

2. Estimates on the Bergman kernel.

Let $\Omega$ be the domain in $\mathbb{C}^n$ considered in Section 1. In this section, we will analyze the local geometry of the domain $\Omega$ near $z_0 \in b\Omega$. We may assume that there are coordinate functions $z_1, \ldots, z_n$ defined near $z_0$ such that $|\langle \partial r/\partial z_1 \rangle(z)| \geq c$ for all $z$ in a neighborhood $U$ of $z_0$, for some $c > 0$. Let us fix $z' \in U$ for a moment. After an affine transformation for the coordinates $z_2, \ldots, z_{n-1}$, we have coordinate functions $w_1, w_2, \ldots, w_n$ such that $\partial \bar{\partial} r(z') (\partial/\partial w_i, \partial/\partial w_j), 2 \leq i, j \leq n - 1$, is an identity matrix. Then the following special coordinates can be defined by a biholomorphic map $\Phi_{z'}$.

Proposition 2.1 [1, Proposition 2.2]. For each $z' \in U$ and positive even integer $m$, there is a biholomorphic map $\Phi_{z'} : \mathbb{C}^n \to \mathbb{C}^n, \Phi_{z'}^{-1}(z') = 0, \Phi_{z'}^{-1}(z) = \cdots = \Phi_{z'}^{-1}(z) = \cdots = \Phi_{z'}^{-1}(z)$.
$(\zeta_1, \ldots, \zeta_n)$ such that

$$r(\Phi_{\zeta'}(\zeta)) = r(z') + \Re \zeta_1 + \sum_{\alpha=2}^{n-1} \sum_{j+k \leq m/2}^{\text{Re}} \left( b_{j,k}(z') \zeta_1^{j} \zeta_n^{k} \zeta_\alpha \right)$$

$$+ \sum_{j+k \leq m}^{\text{Re}} a_{j,k}(z') \zeta_1^{j} \zeta_n^{k} + \sum_{\alpha=2}^{n-1} |\zeta_\alpha|^2$$

$$+ \mathcal{O}(|\zeta_1||\zeta| + |\zeta''| |\zeta| + |\zeta''| |\zeta_n|^m/2 + 1 + |\zeta_\alpha|^m) .$$

Set $\rho(\zeta) = r \circ \Phi_{\zeta'}(\zeta)$, and set

$$A_l(z') = \max \{|\partial^l \rho| / \partial \zeta_l \zeta_n (0); j = k \},\ 2 \leq l \leq m, \text{ and}$$

$$B_l(z') = \max \{|\partial^{l+1} \rho| / \partial \zeta_l \zeta_n \partial \zeta_\alpha (0); j = k \},\ 2 \leq l' \leq m/2.$$ 

For each $\delta > 0$, we define $\tau(z', \delta)$ as follows

$$\tau(z', \delta) = \min \{ (\frac{\delta}{A_l(z')})^{1/l}, (\frac{\delta^{1/2}}{B_l(z')})^{1/l'} \},$$

In [1], it was shown that $(\delta^{1/2} / B_l(z'))^{1/l'} \gg \tau(z', \delta)$ whenever $\delta > 0$ is sufficiently small. Hence the terms mixed with $\zeta_n$ and $\zeta_\alpha, \alpha = 2, \ldots, n - 1$, would not be an important ones in (2.1) and hence

$$\tau(z', \delta) = \min \{ (\frac{\delta}{A_l(z')})^{1/l};\ 2 \leq l \leq m \} .$$

Since $A_m(z_0) \geq c > 0$, it follows that $A_m(z') \geq c' > 0$ for all $z' \in U$ if $U$ is sufficiently small. This gives the inequality

$$\delta^{1/2} \lesssim \tau(z', \delta) \lesssim \delta^{1/m},\ z' \in U,$$

and the definition of $\tau(z', \delta)$ easily implies that if $\delta' < \delta''$, then

$$\tau(\delta'/\delta'')^{1/2} \tau(z', \delta'') \leq \tau(z', \delta') \leq (\delta'/\delta'')^{1/2} \tau(z', \delta'').$$

Now set $\tau_1 = \delta, \tau_2 = \ldots = \tau_{n-1} = \delta^{1/2}, \ \tau_n = \tau(z', \delta) = \tau$ and define

$$R_{\delta}(z') = \{ \zeta \in \mathbb{C}^n; |\zeta_k| < \tau_k,\ k = 1, 2, \ldots, n \}, \text{ and}$$

$$Q_{\delta}(z') = \{ \Phi_{z'}(\zeta); \zeta \in R_{\delta}(z') \} .$$

In the sequel we denote any partial derivative operator of the form $\partial^{\mu+\nu} / \partial \zeta_k^{\mu} \zeta_k^{\nu}$ by $D_k^{\mu+\nu}$, where $\mu + \nu = l, k = 1, 2, \ldots, n$. By the definitions of $\tau_k, k \geq 1$, one has the following useful derivative estimates for the function $\rho = r \circ \Phi_{z'}$. 

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Proposition 2.2 [1, Proposition 2.3]. Let \( z' \in U \). Then the function \( \rho = r \circ \Phi_{z'}(\zeta) \) satisfies

\[
|\rho(\zeta) - \rho(0)| \lesssim \delta, \quad \zeta \in R_\delta(z'),
\]
and

\[
|D_k^l D_n^i \rho(\zeta)| \lesssim \delta \tau_n^{-l} \tau_k^{-i}, \quad \zeta \in R_\delta(z'),
\]

for \( l + im/2 \leq m, \ i = 0, 1, k = 2, \ldots, n-1 \).

In [1], the author proved that for \( z \in Q_\delta(z') \)

(2.6)

\[
\tau(z', \delta) \approx \tau(z, \delta).
\]

Now let us study how the polydiscs \( Q_\delta(z') \) and \( Q_\delta(z'') \) are related. Let \( \Phi_{z'} \) be the map as in Proposition 2.1, and set \( \Phi_{z'}(\zeta'') = z'' \). If we apply Proposition 2.1 at the point \( \zeta'' \) with \( r \) replaced by \( \rho = r \circ \Phi_{z'} \), then we obtain a map \( \Psi_{z''} : \mathbb{C}^n \to \mathbb{C}^n \).

By virtue of the proof of Proposition 2.1 ([1, Proposition 2.2]), we see that \( \Psi_{z''} = \phi^1 \circ \phi^2 \circ \ldots \circ \phi^m \), where for \( l \geq 2 \) and \( \rho_l = \rho \circ \phi^1 \circ \ldots \circ \phi^{l-1} \),

\[
\phi^l(u) = (\phi^l_1(u), \ldots, \phi^l_n(u)) = (\zeta_1, \ldots, \zeta_n)
\]

is a biholomorphic map on \( \mathbb{C}^n \) given by

\[
u_1 = z_1 + \frac{2}{l!} \frac{\partial^l \rho_0(0)}{\partial z_n^l} z_n^l + \frac{2}{l!} \sum_{\alpha=2}^{n-1} \frac{\partial^{l+1} \rho_0(0)}{\partial z_n^l \partial z_i^\alpha} z_\alpha z_n^l,
\]

\[
u_j = z_j, \quad j = 2, \ldots, n,
\]

followed by the coordinate change

\[
z_1 = \zeta_1, \quad z_n = \zeta_n, \quad z_\alpha = \zeta_\alpha + \frac{\partial^{l+1} \rho(0)}{\partial z_n^l \partial z_n^\alpha} \zeta^l,
\]

and \( \phi^1 \) is an affine transformation which is uniformly non-singular in \( U \). From Proposition 2.2, \( \phi^2 \) satisfies, for \( l + im/2 \leq m, \ i = 0, 1, k = 2, \ldots, n-1 \), that

(2.7)

\[
|D_k^l D_n^i \phi^2_0(0)| \lesssim \delta \tau_n^{-l} \tau_k^{-i} \quad \text{and}
\]

\[
|D_k^l D_n^i \phi^2_\alpha(0)| \lesssim \delta^{1/2} \tau_n^{-l} \tau_k^{-i}, \quad \alpha = 2, \ldots, n-1.
\]

By induction, one can show that the same estimates hold for the components of \( \phi^l \). Since \( \Psi_{z''} = (\psi_1, \ldots, \psi_n) \) is a composite of \( \phi^l, l = 1, \ldots, m \), and since each \( \phi^l \) satisfies an analog of (2.7), we have the following estimates for the component functions \( \psi_k \) of \( \Psi_{z''} \).

Lemma 2.3. For \( l + im/2 \leq m, \ i = 0, 1, k = 2, \ldots, n-1 \), one has

(2.8)

\[
|D_k^l D_n^i \psi_0(0)| \lesssim \delta \tau_n^{-l} \tau_k^{-i} \quad \text{and}
\]

\[
|D_k^l D_n^i \psi_\alpha(0)| \lesssim \delta^{1/2} \tau_n^{-l} \tau_k^{-i}, \quad \alpha = 2, \ldots, n-1.
\]
Remark 2.4. Since the component functions of $\Psi_{z''}^{-1}$ have expressions similar to those of $\Psi_{z''}$, they satisfy the same estimates as (2.8).

Proposition 2.5. There exists a constant $C$ such that if $z'' \in Q_\delta(z')$, then

\begin{align}
(2.9) & \quad Q_\delta(z'') \subset Q_{C\delta}(z') \quad \text{and} \\
(2.10) & \quad Q_\delta(z') \subset Q_{C\delta}(z'')
\end{align}

Proof. Define $S_\delta(z'') = \{\Psi_{z''}(u); u \in R_\delta(z'')\}$. From (2.5) we see that to prove (2.9), it suffices to show that

$$S_\delta(z'') \subset R_{C\delta}(z').$$

Note that (2.6) implies that $\tau(z'', \delta) \lesssim \tau(z', \delta)$. Since $\zeta'' = (\Phi_{z''})^{-1}(z'') \in R_\delta(z')$, it follows that if $\zeta \in S_\delta(z'')$, then

$$|\zeta_n| = |\zeta''_n + u_n| < |\zeta''_n| + \tau(z'', \delta) \lesssim \tau(z', \delta) + \tau(z'', \delta) \lesssim \tau(z', \delta),$$

where we have used the fact that $u \in R_\delta(z'')$, and hence that $|u_n| \lesssim \tau(z'', \delta)$. Also by Lemma 2.3 and by the Taylor series expansion theorem,

$$|\zeta_\alpha| = |\psi_\alpha(u)| = |\zeta''_\alpha + \psi_\alpha(u)| \lesssim \delta^{1/2} + \sum_{1 \leq k \leq m/2} \delta^{1/2} \tau^{-k} |u_n|^k \lesssim \delta^{1/2},$$

for $\alpha = 2, \ldots, n - 1$, and

$$|\zeta_1| = |\zeta''_1 + \psi_1(u)| \lesssim \delta + |\psi_1(u)| \lesssim \delta + \delta \lesssim \delta.$$

This shows that $\zeta \in R_{C\delta}(z')$ and proves (2.9). To prove (2.10), define $\tilde{R}_\delta(z') = \{\Psi_{z''}^{-1}(\zeta); \zeta \in R_\delta(z')\}$. Then (2.5) also implies that it is sufficient to prove that

$$\tilde{R}_\delta(z') \subset R_{C\delta}(z'').$$

Since each component function of $\Psi_{z''}^{-1}$ also satisfies the same estimates as that of $\Psi_{z''}$ and since $\tau(z', \delta) \lesssim \tau(z'', \delta)$, we may apply the same method as above to prove $\tilde{R}_\delta(z') \subset R_{C\delta}(z'')$. \qed

For $z^1$ and $z^2$ in $U \cap \Omega$, let $\Phi_{z^1}$ be the biholomorphic map as in Proposition 2.1 associated with $z^1$ and set $0 = \zeta^1 = \Phi_{z^1}^{-1}(z^1)$, $\zeta^2 = \Phi_{z^2}^{-1}(z^2)$. Then we define

$$d_1(z^1, z^2) = \inf\{\eta > 0; z^2 \in Q_\eta(z^1)\},$$

and set

$$M_1(z^1, z^2) = |\zeta_1^1 - \zeta_2^2| + \sum_{j=2}^{n-1} |\zeta_j^1 - \zeta_j^2|^2 + \sum_{l=2}^{m} A_l(z^1) |\zeta_n^1 - \zeta_n^2| = M(\zeta^1, \zeta^2).$$

Then from the definitions (2.2), (2.3), (2.5), and by virtue of the proof of Proposition 2.1, we have

$$d_1(z^1, z^2) \approx M_1(z^1, z^2).$$
Proposition 2.6. \( d_1(z^1, z^2) \) is a pseudometric on \( U \cap \Omega \).

Proof. Suppose \( d_1(z^1, z^2) \neq 0 \) and choose \( \alpha > d_1(z^1, z^2) \). Then \( Q_\alpha(z^1) \cap Q_\alpha(z^2) \) is not empty and by Proposition 2.5, \( Q_\alpha(z^1) \subset Q_C(z^2) \) for an independent constant \( C \). Thus \( z^1 \in Q_C(z^2) \) for all \( \alpha > d_1(z^1, z^2) \). It follows that \( d_1(z^1, z^2) \leq C d_1(z^2, z^1) \). Let \( z^1, z^2, z^3 \in U \cap \Omega \) and set \( \beta = \max\{d_1(z^1, z^2), d_1(z^3, z^2)\} \). Then \( Q_\beta(z^1) \cap Q_\beta(z^3) \neq \emptyset \), and hence Proposition 2.5 implies that \( Q_\beta(z^3) \subset Q_C(z^1) \) for an independent constant \( C \). It thus follows that

\[
d_1(z^1, z^3) \leq C \beta \leq C(d_1(z^1, z^2) + d_1(z^3, z^2))) \leq C^2(d_1(z^1, z^2) + d_1(z^2, z^3)).
\]

\( \square \)

We recall the estimates on the Bergman kernel function and its derivatives for the domain \( \Omega \) obtained in [1], [2].

Theorem 2.7. Let \( \Omega \) and \( z_0 \in b\Omega \) be as above. For \( z^1, z^2 \in U \cap \Omega \), set \( \zeta^i = \Phi^{-1}_z(z^i) \), \( i = 1, 2 \). Then there exist a neighborhood \( U \) of \( z_0 \) and constants \( C_{\alpha, \beta} \), independent of \( z^1, z^2 \in U \cap \Omega \), such that

\[
|D_{\zeta^1}^a \overline{D}_{\zeta^2}^b K_{\zeta^1, \zeta^2} (\zeta^1, \zeta^2)| \leq C_{\alpha, \beta}\delta^{-n-\alpha_1-\beta_1-(\alpha_2+\beta_2+\cdots+\alpha_n+\beta_n)/2} \tau(z^1, \delta)^{-2-n-\alpha-n-\beta} \]

where \( \delta = |\rho(\zeta^1)| + |\rho(\zeta^2)| + M(\zeta^1, \zeta^2) \), and \( \rho = r \circ \Phi_z \).

3. \( L^p \)-boundedness of the Bergman projection.

Now we construct a global pseudometric \( d \) based simply on patching together the local pseudometric \( d_1(z', z) \). Let \( B_j = B(a^j; \epsilon/2), j = 1, \ldots, N \), be a minimal open covering of \( b\Omega \) by ordinary Euclidean balls with centers \( a^j \in b\Omega \) and radius \( \epsilon/2 > 0 \) such that \( B(a^j; 2\epsilon), j = 1, 2, \ldots, N \), are the set of the neighborhoods given by Theorem 2.7 and \( B(a^j; \epsilon/4) \cap B(a^k; \epsilon/4) = \emptyset \) for all \( j \neq k \). Set \( B_0 = \Omega - \bigcup_{j=1}^N B_j \) and let \( d_j(z', z) \) be defined on \( B(a^j; 2\epsilon) \) by (2.11). Choose \( \phi_j \in C_0^\infty(B(a^j; 2\epsilon)) \), \( \phi_j \geq 0, j = 1, \ldots, N \), with \( \phi_j(z) = 1 \) if \( z \in B(a^j; 3\epsilon/2) \), and set

\[
d_0(z', z) = \sum_{j=1}^N \phi_j(z') \phi_j(z) d_j(z', z).
\]

Then \( d_0 \) is well-defined by the compatibility of the functions \( d_j(z', z) \) on the overlaps of the covering, that is, \( d_j(z', z) \approx d_k(z', z) \) if \( z', z \in B(a^j; \epsilon/2) \cap B(a^k; \epsilon/2) \).

To obtain a global pseudometric on \( \Omega \), set

\[
d(z', z) = \begin{cases} d_0(z', z), & |z' - z| < \epsilon \\ |z' - z|, & \text{otherwise}. \end{cases}
\]

Then it is easy to show that \( d \) is a pseudometric on \( \Omega \).

Lemma 3.1. Let \( \mu \) be the Lebesgue measure on \( \Omega \). Then the triple \((\Omega, d, \mu)\) is a space of homogeneous type.

Proof.

\[
\mu(Q_\delta(z')) \approx \prod_{j=1}^n \tau_j(z', \delta)^2 = \delta^n \tau(z', \delta)^2 < \infty
\]
and
\[
\mu(Q_{2\delta}(z')) \approx \prod_{j=1}^{n} \tau_j(z', 2\delta)^2 = (2\delta)^n \tau(z', 2\delta)^2 \lesssim 2^{n+1} \delta^n \tau(z', \delta)^2 = 2^{n+1} \mu(Q_{\delta}(z', \delta)).
\]

For small \(\delta\) and \(z\) near \(b\Omega\), the volume of the balls \(P(z', \delta) = \{z : d(z', z) < \delta\}\) are comparable with those of the polydiscs \(Q_{\delta}(z')\). Thus it follows that
\[
\mu(P(z', \delta)) < \infty
\]
and
\[
\mu(P(z', 2\delta)) \leq C \mu(P(z', \delta)),
\]
where \(C\) is an independent constant. Thus \((\Omega, d, \mu)\) is a space of homogeneous type. \(\square\)

Now assume that \(z', z, w \in B(a_j; \varepsilon) \cap \Omega\) for some \(j\) and consider the biholomorphic map \(\Phi_{z'}\) as in Proposition 2.1 and set \(\zeta' = 0 = \Phi_{z'}^{-1}(z')\), \(\zeta = \Phi_{z'}^{-1}(z)\), \(\xi = \Phi_{z'}^{-1}(w)\). Note that \(d(z', z) \approx M(\zeta', \zeta) \approx d_j(z', z)\) in this case.

**Lemma 3.2.** Let \(\zeta', \zeta, \xi\) be given as above. Then there are \(\nu > 0\), and \(T > 0\) such that
\[
|K(\zeta', \zeta) - K(\xi, \zeta)| \lesssim \left( \frac{M(\zeta', \xi)}{M(\zeta', \zeta)} \right)^{\nu} \frac{1}{\text{Vol}(P_{M(\zeta', \xi)}(\zeta'))},
\]
for \(M(\zeta', \zeta) > TM(\zeta', \zeta)\). Here \(K = K_{\Omega_j}\), and \(P_{\delta}(\zeta') = \{\zeta : M(\zeta', \zeta) < \delta\}\).

**Proof.** From the definitions of (2.11) and (2.12), we have \(\xi \in P_{2M(\zeta', \xi)}(\zeta')\). If we apply Proposition 2.1 at the point \(\xi'\) with \(r\) replaced by \(\rho = r \circ \Phi_{z'}\) and by virtue of Theorem 2.7, it follows that
\[
|K(\zeta', \zeta) - K(\xi, \zeta)| \lesssim \sum_{j=1}^{n} \left| \frac{\partial}{\partial z_j} K(\xi', \zeta) \right| \left| \zeta'_j - \zeta_j \right|
\]
\[
\lesssim \left( \sum_{j=1}^{n} \frac{|\zeta'_j - \zeta_j|}{\tau_j(\xi', M(\zeta', \zeta))} \right) \cdot \frac{1}{\text{Vol}(P_{M(\zeta', \xi)}(\zeta'))},
\]
for some \(\xi' \in P_{C_{M(\zeta', \zeta)}(\zeta')} \cap \Omega\). It follows from the definition of \(M(\zeta^1, \zeta^2)\) and (2.6) that \(M(\zeta', \zeta) \approx M(\zeta', \xi)\) for \(\zeta\) satisfying \(TM(\zeta', \xi) < M(\zeta', \zeta)\), provided \(T\) is sufficiently large. Note that \(P_{\delta}(\zeta')\) and \(R_{\delta}(z')\) (as in (2.5)) are comparable in the sense that \(P_{\delta/C}(\zeta') \subset R_{\delta}(z') \subset P_{C\delta}(\zeta')\) for an independent constant \(C\). Thus \(R_{C^2M(\zeta', \xi)}(\zeta') \cap R_{C^{2M(\zeta', \xi)}(\zeta')} \neq \emptyset\) and hence \(R_{C^{2M(\zeta', \xi)}(\zeta')} \subset R_{C^M(\zeta', \xi)}(\zeta')\) by Proposition 2.5 for an independent constant \(C\). Therefore
\[
\text{Vol}(P_{M(\zeta', \xi)}(\zeta')) \lesssim \text{Vol}(P_{M(\zeta', \xi)}(\xi')) \approx \text{Vol}(P_{M(\zeta', \xi)}(\zeta')).
\]
Set \(\kappa = \min \{k : (M(\zeta', \zeta)/A_k(\zeta'))^{1/k} = \tau_n(\zeta', M(\zeta', \zeta))\}\) where \(\tau_n\) is defined as in (2.3) at \(\zeta'\) with \(r\) replaced by \(r \circ \Phi_{z'}\). Since \(\zeta', \zeta' \in P_{C_{M(\zeta', \zeta)}}(\xi') \subset R_{C^{2M(\zeta', z')}}(\zeta')\), it follows from (2.4) and (2.6) that
\[
\tau_n(\zeta', M(\zeta', \zeta)) \approx \tau_n(\zeta', M(\zeta', \xi)) \approx \tau_n(\zeta', M(\zeta', \zeta)).
\]
By virtue of the definitions of $\tau_i$, $i = 1, 2, \ldots, n - 1$, we also have

\begin{equation}
\tau_1(\xi', M(\xi', \zeta)) \approx M(\xi', \zeta) \approx M(\zeta', \zeta), \text{ and} \\
\tau_j(\xi', M(\xi', \zeta)) \approx M(\xi', \zeta)^{1/2} \approx M(\zeta', \zeta)^{1/2}.
\end{equation}

With (3.4) and the definitions of $\kappa$ and $M(\zeta', \xi)$, we have

\begin{equation}
\frac{|\zeta_n' - \xi_n|}{\tau_n(\xi', M(\xi', \zeta))} \approx \frac{|\zeta_n' - \xi_n|}{\tau_n(\zeta', M(\zeta', \zeta))} \approx \frac{A_\kappa(z')^{1/\kappa}|\zeta_n' - \xi_n|}{M(\zeta', \zeta)^{1/\kappa}} \lesssim \left(\frac{M(\zeta', \xi)}{M(\zeta', \zeta)}\right)^{1/\kappa}.
\end{equation}

Because $M(z', \xi)/M(z', \zeta) \leq 1$ and $\kappa \geq 2$, we also have from (3.5) that

\begin{equation}
\frac{|\zeta_j' - \xi_j|}{\tau_j(\xi', M(\xi', \zeta))} \lesssim \left(\frac{M(\zeta', \xi)}{M(\zeta', \zeta)}\right)^{1/\kappa}, \text{ for } j = 1, 2, \ldots, n - 1.
\end{equation}

We get (3.1) if we combine (3.2), (3.3), (3.6) and (3.7). □

**Theorem 3.3.** The Bergman kernel function $K_\Omega(z', z)$ associated to the domain $\Omega$ is a standard kernel with respect to the metric $d$ and the Lebesgue measure.

**Proof.** Let $\mu$ denote the Lebesgue measure. Minimally cover $\overline{\Omega}$ by open balls $B(a^j; \epsilon/2), j = 0, 1, \ldots, N$, as in the beginning of this section. Since $K_\Omega(z', z)$ is smooth away from the boundary diagonal, there is a constant $C$ such that $|K(z', z)| \leq C$ if $z', z \in B_0$ or if $z' \in B(a^j; \epsilon/4)$ and $z \in B(a^k; \epsilon/4)$ for some $j \neq k$. Also, in this case, $(\mu(P(z', d(z', z))))^{-1} \leq C'$. Thus it follows that

$$|K(z', z)| \lesssim \frac{1}{\mu(P(z', d(z', z)))}.$$

Now assume that $z', z \in B(a^j; \epsilon)$. Then, by Theorem 2.7 and the transformation formula for the Bergman kernel function, it follows that

$$|K(z', z)| \lesssim (d(z', z))^{-n}(\tau(z', d(z', z)))^{-2}$$

$$= \prod_{j=1}^{n} \tau_j(z', d(z', z))^{-2} \approx \frac{1}{\mu(P(z', d(z', z)))},$$

where $d(z', z)$ is associated to $B(a^j; \epsilon)$. Thus for any $z', z \in \Omega$,

$$|K(z', z)| \lesssim \frac{1}{\mu(P(z', d(z', z)))}.$$

Assume now that $z', w \in \Omega$ with $z' \neq w$. Then

$$\int_{d(z', z) > Td(z', w)} |K(z', z) - K(w, z)|dz = \int_{A_T'} |K(z', z) - K(w, z)|dz$$

$$+ \int_{B_T'} |K(z', z) - K(w, z)|dz$$

$$:= I + II,$$
where $A_T^T(z) = \{z : d(z', z) > Td(z', w), \ |z' - z| \geq \epsilon/2\} \cap \Omega$ and $B_T^T(z) = \{z : d(z', z) > Td(z', w), \ |z' - z| \leq \epsilon/2\} \cap \Omega$. Note that $|z - w| \geq \epsilon/4$ for $z \in A_T^T(z)$ provided $T$ is sufficiently large. Since $K(\cdot, \cdot)$ is smooth away from the boundary diagonal, it follows that $I \lesssim 1$. To estimate $II$, assume that $z' \in B(a^j; \epsilon/2)$ for some $j$. Thus $z', z, w \in B(a^j; \epsilon)$ for $z \in B_T^T$. Let $\Phi_{z'}$ be the biholomorphic map associated with $z'$ as in Proposition 2.1 and set $\zeta = \Phi_{z'}^{-1}(z)$, $\xi = \Phi_{z'}^{-1}(w)$. Then

$$II = \int_{B(a^j; \epsilon) \cap B_T^T} |K_\Omega(z', z) - K_\Omega(w, z)|dz$$

$$\lesssim \int_{\{\zeta : M(0, \zeta) > TM(0, \zeta)\}} |K_{\Omega_z}(0, \zeta) - K_{\Omega_{z'}}(\xi, \xi)|d\zeta := III.$$

Set $\delta = M(0, \xi)$ and define the dyadic rings $D_k = \{\zeta : 2^k(T\delta) < M(0, \zeta) < 2^{k+1}(T\delta)\}$. Recall that $R_\delta(z') \approx P(\zeta', \delta) = \{\zeta : M(\zeta', \zeta) < \delta\}$. Thus it follows from Lemma 3.2 and the “doubling property” of the ball that

$$III \lesssim \int_{\cup D_k} \left( \frac{M(0, \zeta)}{M(0, \xi)} \right)^\nu \cdot \frac{1}{\Vol(P(0, M(0, \zeta)))} d\zeta$$

$$= \delta^\nu \sum_{k=0}^\infty \int_{D_k} M(0, \zeta)^{-\nu} \cdot \frac{1}{\Vol(P(0, M(0, \zeta)))}$$

$$\leq \sum_{k=0}^\infty \frac{\delta^\nu}{(2^k T\delta)^\nu} \cdot \frac{\Vol(P(0, 2^{k+1}T\delta))}{\Vol(P(0, 2^k T\delta))} \lesssim 1.$$

\[\square\]

**Definition 3.4.** Let $(X, \mu)$ be a measure space. An operator $T : L^p(X, \mu) \to \{\text{measurable functions on } X\}$ is said to be of weak type $(p, p)$, $0 < p < \infty$, if

$$\mu\{x; |Tf(x)| > \lambda\} \leq C \frac{\|f\|_p^p}{\lambda^p}, \text{ all } f \in L^p, \lambda > 0,$$

where $C$ is a constant independent of $f$ and $\lambda$.

If we use Theorem 3.3 and the Calderón- Zygmund decomposition of $\Omega$ in terms of balls $P(\zeta', \delta)$ (this is analogous to the Calderón-Zygmund decomposition of $\mathbb{R}^n$ in terms of standard cubes (see [3])), it is a routine matter to show that the Bergman kernel is of weak type $(1,1)$ on $\Omega$ (cf. [3],[5]), and we get the following corollary:

**Corollary 3.5.** The Bergman projection $P$ is of weak type $(1,1)$ on $\Omega$.

If we combine Theorem 3.3, Corollary 3.5 and the $L^2$-boundedness of $P$, it follows that the following theorem holds.

**Theorem 3.6.** Let $P$ be the Bergman projection associated to the domain $\Omega$ in Section 1. Then $P$ maps $L^p(\Omega)$ to $L^p(\Omega)$, boundedly, for all $1 < p < \infty$.

**References**


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