ON THE MERGELEYAN APPROXIMATION
PROPERTY ON PSEUDOCONVEX DOMAINS IN $\mathbb{C}^n$

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Abstract. Let $\Omega$ be a smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^n$.
We prove the Mergelyan approximation property in various topologies on $\Omega$ where
the estimates for $\overline{\partial}$-equation are known in the corresponding topologies.

1. Introduction.

Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ and let $H(\Omega)$ denote the functions holomorphic
on $\Omega$. We say $\Omega$ has the Mergelyan property if every $f \in H(\Omega) \cap C(\overline{\Omega})$ can
be approximated uniformly on $\overline{\Omega}$ by functions in $H(\Omega)$, where $H(\Omega)$ denote the
functions holomorphic in a neighborhood of $\overline{\Omega}$.

For $n > 1$, Henkin [9], Kerzman [10] and Lieb [11] proved the Mergelyan property
on strongly pseudoconvex domains in $\mathbb{C}^n$. The key technical tools to prove the
Mergelyan property were:

(1) existence of Stein neighborhood basis for $\overline{\Omega}$, and
(2) Lipschitz estimates and their stability for $\overline{\partial}$ on these Stein neighborhood
basis.

Notice that the author has constructed a Stein neighborhood basis of $\overline{\Omega}$ when $\Omega$ is
a smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^n$ [3].

Let $\Lambda^\alpha(\Omega)$ denote the usual Lipschitz space of order $\alpha \geq 0$ with norm $\| \cdot \|_{\Lambda^\alpha(\Omega)}$
and let $L^p_k(\Omega)$ denote the space of functions on $\Omega$ that are in $L^p(\Omega)$ along with all
their derivatives up to order $k$, with norm denoted by $\| \cdot \|_{L^p_k(\Omega)}$. Lipschitz estimates
for $\overline{\partial}$ on pseudoconvex domains in $\mathbb{C}^n$ are known for some special kinds of domains.
That is, smoothly bounded pseudoconvex domains of finite type in $\mathbb{C}^2$ [2,7], and
smoothly bounded pseudoconvex domain $\Omega$ of finite type in $\mathbb{C}^n$ such that the Levi-
form of $\partial \Omega$ is diagonalizable [8], etc. However it is difficult and sometimes tedious
matter to prove the stability of the estimates for $\overline{\partial}$ even though the estimates are
known [6,12].

In this paper, we will present a new method to prove the Mergelyan property in
various topologies. That is, if $\Omega$ has a Stein neighborhood basis with the estimates
for $\overline{\partial}$ in $\Lambda^\alpha(\Omega)$, $\alpha \geq 0$ or in $L^p_k(\Omega)$, $1 < p < \infty$, $k \geq 0$, we will prove the Mergelyan
property in the corresponding topologies. Here we are not assuming the stability
of the estimates for $\overline{\partial}$ in the corresponding spaces. Nevertheless, we will use the known result of the stability of estimates for $\overline{\partial}$ in $L^2_k(\Omega)$ spaces [4].

We will state and prove our results only on smoothly bounded pseudoconvex domains of finite type in $\mathbb{C}^2$. The same (or similar) results hold for the domains of finite type in $\mathbb{C}^n$ with estimates for $\overline{\partial}$ in the corresponding spaces.

**Theorem 1.** Let $\Omega$ be a smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^2$. Assume $f \in H(\Omega) \cap L^p_k(\Omega)$, where $1 < p < \infty$ and $k$ is a non-negative integer. Then there is a sequence $\{g_n\} \subset H(\Omega)$ such that

$$\lim_{n \to \infty} \|g_n - f\|_{L^p_k(\Omega)} = 0.$$

Let $\mathbb{N}$ denote the set of natural numbers.

**Theorem 2.** Let $\Omega$ be as in Theorem 1 and assume $f \in H(\Omega) \cap \Lambda^\alpha(\Omega)$. Then for each $\alpha' < \alpha$ (with $\alpha \in \{0\} \cup \mathbb{N}$) arbitrary given, there is a sequence $\{g_n\} \subset H(\overline{\Omega})$ such that

$$\lim_{n \to \infty} \|g_n - f\|_{\Lambda^\alpha'(\Omega)} = 0.$$

**Remark 3.** In [5], the author proved that every $f \in H(\Omega) \cap L^2_k(\Omega)$ can be approximated by functions in $H(\Omega)$ in $L^p_k(\Omega)$ topologies, $k \geq 0$, when $\Omega$ is a smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^n$. When $n = 2$, Cho and others [6] also proved the Mergelyan property in Lipschitz spaces $\Lambda^\alpha(\Omega)$, $0 \leq \alpha < 1/m$, for $f \in H(\Omega) \cap \Lambda^\alpha(\Omega)$. Here $m$ is the type of $b\Omega$. Both of these results depend on the solvability and stability of the estimates for $\overline{\partial}$ in the corresponding topologies [4,6].

The key ingredients to prove Theorem 1 and Theorem 2 are the $L^p_k(\Omega)$ and Lipschitz estimates for $\overline{\partial}$ on $\Omega \subset \subset \mathbb{C}^2$ [2,7,8], and the smooth bumping theorem for pseudoconvex domains of finite type in $\mathbb{C}^n$ [2,3,4].

2. Approximation by smooth functions.

In this section, we prove that any holomorphic function in $L^p_k(\Omega)$ (resp. $\Lambda^\alpha(\Omega)$) can be approximated by smooth functions on $\overline{\Omega}$ in appropriate topologies.

Let $U_j$, $j = 0, 1, \ldots, N$ be a finite collection of open sets with the following properties:

(a) $\overline{\Omega} \subset \bigcup_{j=0}^{N} U_j$

(b) $U_0 \subset \subset \Omega$

(c) On each $U_j$, $j = 1, 2, \ldots, N$, there are holomorphic coordinates $z_1^j$, $z_2^j$ with the property that $\partial r / \partial x_2^j > 0$, where $z_2^j = x_2^j + iy_2^j$.

Let $\zeta_j$, $j = 0, 1, \ldots, N$ be a partition of unity subordinate to the covering $\{U_j\}$. For all sufficiently small $\delta > 0$, and for a given function $f \in H(\Omega)$, let $f_\delta$ be given by

$$f_\delta(z) = \zeta_0(z)f(z) + \sum_{j=1}^{N} \zeta_j(z)f(z_1^j, z_2^j - \delta).$$

Observe that $f_\delta \in C^\infty(\overline{\Omega})$. 

$$\lim_{n \to \infty} \|g_n - f\|_{L^p_k(\Omega)} = 0.$$
 Proposition 4. Suppose that \( f \in H(\Omega) \cap L^p_k(\Omega), \ 1 < p < \infty, \ k \geq 0 \). Then \( \| f_\delta - f \|_{L^p_k(\Omega)} \rightarrow 0 \) and \( \| \partial f_\delta \|_{L^p_k(\Omega)} \rightarrow 0 \) as \( \delta \rightarrow 0 \).

 Proof. Note that \( D^\alpha(f - f_\delta) \in L^p(\Omega), \ |\alpha| \leq k \). So \( \| f_\delta - f \|_{L^p_k(\Omega)} \) converges to zero as \( \delta \) does by Lebesgue dominated convergence theorem. Also note that

\[
D^\alpha(\partial f_\delta) = D^\alpha(\partial (f_\delta - f)) = D^\alpha \left( \sum_{j=1}^N (\partial \zeta_j)(f(z^j_1, z^j_2 - \delta) - f(z^j_1, z^j_2)) \right).
\]

So \( \| \partial f_\delta \|_{L^p_k(\Omega)} \rightarrow 0 \) as \( \delta \rightarrow 0 \) by the same reasoning. \( \square \)

The functions \( f_\delta \in C^\infty(\Omega) \) defined in (1) also approximate \( f \) in \( \Lambda^\alpha(\Omega) \)-topology:

 Proposition 5. Suppose that \( f \in H(\Omega) \cap \Lambda^\alpha(\Omega) \). Then for each \( \alpha'< \alpha \) (\( \alpha' = \alpha \) if \( \alpha \in \{0\} \cup \mathbb{N} \)), \( \| f_\delta - f \|_{\Lambda^{\alpha'}(\Omega)} \) and \( \| \partial f_\delta \|_{\Lambda^{\alpha'}(\Omega)} \) converges to zero as \( \delta \) does.

 Proof. It is clear that the conclusion holds if \( \alpha \in \{0\} \cup \mathbb{N} \). Without loss of generality, we may assume that \( 0 < \alpha' < \alpha < 1 \). If \( x, y \) satisfy \( \delta \leq |x - y| \), we use the Lipschitz continuity condition on \( (f_\delta - f)(x) \) and \( (f_\delta - f)(y) \), and if \( \delta > |x - y| \), we use the same condition on \( f(x) - f(y) \) and \( f_\delta(x) - f_\delta(y) \). In both cases, we will get \( \| f_\delta - f \|_{\Lambda^{\alpha'}(\Omega)} \leq \delta^{\alpha - \alpha'} \). Since \( \alpha' < \alpha \), this proves that \( f_\delta - f \|_{\Lambda^{\alpha'}(\Omega)} \) converges to zero as \( \delta \) does. Similarly, we can prove that \( \| \partial f_\delta \|_{\Lambda^{\alpha'}(\Omega)} \) converges to zero as \( \delta \) does. \( \square \)

 Remark 6. If \( \alpha \not\in \mathbb{N} \cup \{0\} \), the functions \( f_\delta \) defined in (1) does not converge to \( f \) in \( \Lambda^\alpha(\Omega) \) norm in general.

 Definition 7. Let \( \Omega \subset \mathbb{C}^n \) be a smoothly bounded pseudoconvex domain with \( C^\infty \) defining function \( r \). By a smooth bumping family of \( \Omega \) we mean a family of smoothly bounded pseudoconvex domains \( \{ \Omega_\tau \}_{0 \leq \tau \leq 1} \) satisfying the following properties:

\begin{enumerate}
  \item \( \Omega_0 = \Omega \)
  \item \( \Omega_{\tau_1} \Subset \Omega_{\tau_2} \) if \( \tau_1 < \tau_2 \),
  \item \( \{ b\Omega_\tau \}_{0 \leq \tau \leq 1} \) is a \( C^\infty \) family of real hypersurfaces in \( \mathbb{C}^n \),
  \item the boundary defining functions \( r_\tau \) of \( \Omega_\tau \) varies smoothly with respect to \( \tau \) \( r_\tau \rightarrow r \) as \( \tau \rightarrow 0 \) in \( C^\infty \) topology.
\end{enumerate}

 In [3], the author constructed a smooth bumping family of \( \Omega \) if \( b\Omega \) is of finite type.

 For the final remark of this section, we state the following theorem which says the stability of \( L^2 \)-estimates for \( \partial \) on \( \Omega \) [4].

 Theorem 8. Let \( \{ \Omega_\tau \}_{0 \leq \tau \leq 1} \) be a smooth bumping family of pseudoconvex domains in \( \mathbb{C}^n \). Then for each \( m \) there exists a constant \( C_m \), independent of \( \tau \), such that

\[
\| f_\tau \|_m \leq C_m \| \partial_\tau f_\tau \|_m,
\]

for all \( f_\tau \in \text{Dom}(\square_\tau) \cap C^\infty(\overline{\Omega_\tau}) \) with \( f_\tau \perp H^{0,1}(\overline{\Omega_\tau}) \). Here \( \square_\tau \) denotes the complex Laplacian on \( \Omega_\tau \).
3. Approximation by holomorphic functions.

Let $\Omega$ be a smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^2$ and let $N$ be the Neumann operator associated with the $\bar{\partial}$-Neumann problem. Then the main results in [2,7] show:

\begin{equation}
\bar{\partial}^* N : L^p_k(\Omega) \longrightarrow L^p_k(\Omega), \quad 1 < p < \infty, \quad k \geq 0,
\end{equation}

and

\begin{equation}
\bar{\partial} N : \Lambda^\alpha(\Omega) \longrightarrow \Lambda^\alpha(\Omega) \quad \alpha \geq 0,
\end{equation}

are bounded operators on the corresponding spaces.

Now let us prove Theorem 1 and Theorem 2. Here we will only present a proof of Theorem 2. The proof of Theorem 1 follows the same (or similar) lines. Let us state Theorem 2 again:

**Theorem 2.** Let $\Omega$ be a smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^2$. Assume $f \in H(\Omega) \cap \Lambda^\alpha(\Omega)$. Then for each $\alpha' < \alpha$ ($\alpha' = \alpha$ if $\alpha \in \{0\} \cup \mathbb{N}$) arbitrary given, there is a sequence $\{g_n\} \subset H(\Omega)$ such that

\[ \lim_{n \to \infty} \|g_n - f\|_{\Lambda^{\alpha'}(\Omega)} = 0. \]

**Proof.** Let $f_\delta$ be defined by (1). By Proposition 5, $f_\delta \in C^\infty(\Omega)$ converge to $f$ in $\Lambda^\alpha(\Omega)$ topology as $\delta$ goes to zero. For each $\delta > 0$ fixed, let us solve $\bar{\partial}u_\delta = \bar{\partial}f_\delta$ on $\Omega$. Then by Catlin’s global regularity theorem for $\bar{\partial}$-equation on pseudoconvex domains of finite type in $\mathbb{C}^n$ [1], it follows that $u_\delta \in C^\infty(\Omega)$. Also by (3), $u_\delta$ satisfies

\[ \|u_\delta\|_{\Lambda^{\alpha'}(\Omega)} \leq C_{\alpha}\|\bar{\partial}f_\delta\|_{\Lambda^{\alpha'}(\Omega)}, \]

where $C_{\alpha}$ does not depend on $\delta$. Since $\|\bar{\partial}f_\delta\|_{\Lambda^{\alpha'}(\Omega)}$ converges to zero as $\delta$ does, it follows that $\|u_\delta\|_{\Lambda^{\alpha'}(\Omega)} \to 0$ as $\delta \to 0$. Set

\[ h_\delta = f_\delta - u_\delta. \]

Then $h_\delta \in H(\Omega) \cap C^\infty(\Omega)$ and

\[ \lim_{\delta \to 0} \|h_\delta - f\|_{\Lambda^{\alpha'}(\Omega)} = 0. \]

Let $\{\Omega_\tau\}$ be a smooth pseudoconvex bumping family of $\Omega$. We extend $h_\delta$ to $\Omega_\tau$ and set it is $h_\delta^\tau$ on $\Omega_\tau$. Notice that $\bar{\partial}h_\delta^\tau \equiv 0$ on $\Omega$ and that $\bar{\partial}h_\delta^\tau \in C^\infty(\Omega)$. Hence $\bar{\partial}h_\delta^\tau$ vanishes to infinity order on $\partial\Omega$ as $\tau \to 0$. Set $k = [\alpha] + 4$, where $[\alpha]$ denotes the greatest integer less than or equal to $\alpha$. Let us solve the following $\bar{\partial}$-equation in $L^2_k(\Omega)$ spaces (with weighted estimates of $\bar{\partial}$):

\[ \bar{\partial}p_\delta^\tau = \bar{\partial}h_\delta^\tau \quad \text{on} \quad \Omega_\tau. \]

From Theorem 8 (stability of $L^2_k(\Omega_\tau)$-estimates of $\bar{\partial}$-equation), it follows that

\begin{equation}
\|p_\delta^\tau\|_{L^2_k(\Omega_\tau)} \leq C_{\alpha}\|\bar{\partial}h_\delta^\tau\|_{L^2_k(\Omega_\tau)},
\end{equation}

where $C_{\alpha}$ does not depend on $\tau$. Set

\[ g_\delta^\tau = p_\delta^\tau - h_\delta^\tau \in H(\Omega_\tau). \]

Then the estimates in (4), (5) and the Sobolev embedding theorem imply that

\[ \|g_\delta^\tau - f\|_{\Lambda^{\alpha'}(\Omega)} \to 0 \quad \text{as} \quad \delta, \tau \to 0. \]

This proves Theorem 2.
References


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