A LOWER BOUND ON THE KOBAYASHI METRIC NEAR A POINT OF FINITE TYPE IN $\mathbb{C}^n$.

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Abstract. Let Ω be a bounded domain in $\mathbb{C}^n$ and $b\Omega$ is smooth pseudoconvex near $z_0 \in b\Omega$ of finite type. Then there are constants $c > 0$ and $\epsilon' > 0$ such that the Kobayashi metric, $K_\Omega(z;X)$, satisfies $K_\Omega(z;X) \geq c|X|\delta(z)^{-\epsilon'}$ for all $X \in T_z^{1,0}\mathbb{C}^n$ in a neighborhood of z_0. Here $\delta(z)$ denotes the distance from z to $b\Omega$. As an application, we prove the Hölder continuity of proper holomorphic maps onto pseudoconvex domains.

1. Introduction.

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain in $\mathbb{C}^n$. The purpose of this paper is to study the boundary behavior of the Kobayashi metric, $K_\Omega(z;X)$, for z near a point $z_0 \in b\Omega$ of finite type. Here finite type means finite 1-type in D'Angelo sense. We will discuss the definition of finite type in section 2. Let us remind the reader of the definition of Kobayashi metric. The function $K_\Omega : T^{1,0}\Omega \to \mathbb{R}$ on the holomorphic tangent bundle, given by

$$\begin{aligned} K_\Omega(z;X) &= \inf\{\alpha > 0; \exists f : \triangle \to \Omega \text{ holomorphic with } f(0) = z,\ f'(0) = \alpha^{-1}X\} \\ &= \inf\{r^{-1}; \exists f : \triangle_r \to \Omega \text{ holomorphic with } f(0) = z,\ f'(0) = X\}, \end{aligned}$$

is called the Kobayashi metric of Ω. (Here $\triangle$ denotes the unit disc and $\triangle_r = \{t; |t| < r\}$ in $\mathbb{C}$). For a fixed tangent vector X, we will show that $K_\Omega(z;X)$ goes to infinity as z approaches z_0. Our main result is

Theorem 1. *Let Ω be a bounded domain in $\mathbb{C}^n$ and let $b\Omega$ is smooth pseudoconvex in a neighborhood U of $z_0 \in b\Omega$ of finite type. Then there exist a neighborhood $V \subset U$ of z_0 and constants $c > 0$, $\epsilon' > 0$ so that for all $z \in \Omega \cap V$ and $X \in T_z^{1,0}\mathbb{C}^n$*

$$K_\Omega(z;X) \geq c|X| \cdot \delta(z)^{-\epsilon'}$$

where $\delta(z)$ denotes the distance from z to $b\Omega$.

Remark. *The exponent ϵ' in this theorem will not be the largest possible one.*

As an application of Theorem 1, we can prove the Hölder continuity for a class of proper holomorphic maps. Let $\Omega_1, \Omega_2 \subset\subset \mathbb{C}^n$ be bounded pseudoconvex domains in $\mathbb{C}^n$ and $\Phi : \Omega_1 \to \Omega_2$ be a proper holomorphic map. When Ω_1 satisfies condition R, then the C^∞-extendability of Φ holds [2,11]. If $b\Omega_1$ is of finite type, then this is the case. Then the question is whether Φ can be extended smoothly up to $b\Omega_1$ with information about Ω_2. When Ω_1, Ω_2 are pseudoconvex domains, Henkin has shown that the Hölder continuity of Φ up to $\overline{\Omega}_1$ can be proved by using the boundary behavior of Kobayashi metric near $b\Omega_2$ [12]. In other words, if the infinitesimal Kobayashi metric on Ω_2 grows sufficiently fast near the boundary of Ω_2 (i.e., $K_{\Omega_2}(z;X) \geq |X|d(z, b\Omega_2)^{-\epsilon}$ for some $\epsilon \in (0,1)$), then every proper holomorphic map $\Phi : \Omega_1 \to \Omega_2$ extends to a Hölder-continuous map of $\overline{\Omega}_1$ onto $\overline{\Omega}_2$. This holds in particular if Ω_2 is strictly pseudoconvex or if it is pseudoconvex with real analytic boundary [10,12]. The following corollary is an immediate consequence of Theorem 1.

Corollary 2. *Let Ω_1, Ω_2, be bounded pseudoconvex domains in $\mathbb{C}^n$ with $b\Omega_1$ of class C^2 and $b\Omega_2$ C^∞, and $b\Omega_2$ is of finite type in D'Angelo sense. Then there exists an $\epsilon'' > 0$ such that any proper holomorphic map $\Phi : \Omega_1 \to \Omega_2$ extends to a map $\hat{\Phi} : \overline{\Omega}_1 \to \overline{\Omega}_2$ which is Hölder continuous with exponent ϵ''.*

Proof. Since $b\Omega_2$ is pseudoconvex domain of finite type, we can cover $b\Omega_2$ by a finite number of neighborhoods as in Theorem 1. Therefore there exist constants $c_1 > 0$ and $\epsilon_1 > 0$ such that $K_{\Omega_2}(z; X) \geq c_1 |X||r(z)|^{-\epsilon_1}$ for all $z \in \Omega_2$, $X \in T_z^{1,0}\mathbb{C}^n$. If we follow Diederich and Fornaess' method, which was originated from Henkin, we can get Corollary 2. $\square$

In [10], Diederich and Fornaess proved a result similar to Theorem 1 when $b\Omega$ is real analytic. The key point in their proof is a bumping theorem near a given point $z_0 \in b\Omega$. By a bumping theorem, we mean that, we can push out the boundary of Ω preserving pseudoconvexity near a given boundary point. Here we prove the following bumping theorem which says that we can push out the boundary of Ω with certain rate. In [5], Catlin pushed out the boundary of Ω as far as possible when the domain is in $\mathbb{C}^2$.

Theorem 3. *Let $U \subset \mathbb{C}^n$ be an open neighborhood of $z_0 \in \mathbb{C}^n$ and r a C^∞-function on U such that $dr \neq 0$ everywhere on U, $r(z_0) = 0$ and the hypersurface $S = \{z \in U; r(z) = 0\}$ is pseudoconvex and that the type of z_0 is finite. Then there exist open neighborhoods $V', V'' \subset U$ of z_0, $z_0 \in V' \subset\subset V''$, such that for each $z' \in V' \cap S$, there is a 1-parameter family of C^2 functions $\rho_t(z', \cdot)$ on V'' with the following properties*

(i) $\rho_t(z', z') = 0$,
(ii) $\rho_t(z', z)$ *is C^2 in z for z in V'' and smooth in t for $0 \leq t \leq t_0$, t_0 uniform of z',*
(iii) $\frac{\partial \rho_t}{\partial t}(z', z) \leq 0$,
(iv) *For $z \in V'' - \{z'\}$, $\frac{\partial \rho_t}{\partial t}(z', z) < 0$,*
(v) *The hypersurfaces $S_{t,z'} = \{z; \rho_t(z', z) = 0\}$ are pseudoconvex. In fact, $S_{t,z'}$ is strictly pseudoconvex on $S_{t,z'} \cap (V'' - \{z'\})$.*
(vi) *One has $r > 0$ on $S_{t,z'} \cap (V'' - \{z'\})$.*
(vii) *There is $K > 0$ such that $d(z', z)^K \lesssim_t |\rho_t(z', z)|$ for all $z \in V'' \cap \{z : r(z) \leq 0\}$, where $d(z', z) = |z - z'|^2$ and $\lesssim_t$ depends on t.*

The proof of Theorem 3 can be found in section 3. In section 4, we will derive Theorem 1 from Theorem 3.

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2. Finite 1-type domains in $\mathbb{C}^n$.

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n (n \geq 2)$, with C^∞ boundary defining function r, i.e. $\Omega = \{z \in \mathbb{C}^n : r(z) < 0\}$. In this section we will study finite 1-type domains in $\mathbb{C}^n$.

Let $v(g)$ be the order of vanishing at the origin of a holomorphic function $g(t)$, $t \in \mathbb{C}$. If $f = (g_1, ..., g_n)$ is a holomorphic function, then set $v(f) = \min_{1 \leq i \leq n} v(g_i)$. Let $\mathcal{O}rd(f) = \dfrac{v(r \circ f)}{v(f)}$ where f is a 1-dimensional variety satisfying $f(0) = z_0 \in b\Omega$. We call $\mathcal{O}rd(f)$ by "order of contact" of f. Then set

$$\Delta_1(z_0) = \sup_f \mathcal{O}rd(f), \tag{1}$$

and we call $\Delta_1(z_0)$ as the type of z_0 on $b\Omega$. This type function is not an upper semi-continuous function if $n \geq 3$. In [8], D'Angelo found an upper bound of $\Delta_1(z)$ in a neighborhood of z_0.

Theorem 4. *(D'Angelo; [8, Theorem 5.5]) Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ and $z_0 \in b\Omega$. Suppose that $\Delta_1(z_0) < \infty$. Then there is a neighborhood U of z_0 such that for all $z \in b\Omega \cap U$, we have;*

$$\Delta_1(z) \leq 2^{2-n}\Delta_1(z_0)^{n-1}. \tag{2}$$

In [3], Catlin expressed (2) in a more quantitative form using a family of non-singular 1-dimensional manifolds with decreasing diameter.

Theorem 5. *(Catlin; [3, Theorem 3.4]) Let z_0 be a point in the boundary of a smoothly bounded pseudoconvex domain Ω. Assume that $\Delta_1(z_0) < \infty$. Set $T' = 2^{2-n}\Delta_1(z_0)^{n-1}$. Then for any number $\epsilon > 0$, there exist a constant c_ε and a neighborhood U_ε of z_0 such that for any nonsingular 1-dimensional manifold M_σ of diameter σ contained in U_ε,*

$$\sup\{|r(z)| : z \in M_\sigma\} \geq c_\epsilon \sigma^{T'+\epsilon}. \tag{3}$$

Remark. *Theorem 5 shows that the "order of contact" of a family of 1-dimensional manifolds is less than or equal to $T' + \epsilon$.*

Let us take notations in Theorem 5 with $\epsilon = 1$. Also set $T = [\![2^{2-n}(\Delta_1(z_0))^{n-1} + 1]\!]$, where $[\![x]\!]$ denotes the smallest integer bigger than or equal to x. Therefore we have a neighborhood V of z_0 such that if M_σ is a 1-dimensional complex manifold of diameter σ which passes through a point z' in $V \cap b\Omega$, then

$$\sup\{|r(z)| : z \in M_\sigma\} \geq c\sigma^T. \tag{4}$$

In Theorem 3.5 of [7], the author proved the following theorem which is an implementation of Catlin's construction ([3], Theorem 9.2).

Theorem 6. *Let z_0 be a point in the boundary of a pseudoconvex domain Ω with defining function r, and that satisfies $\Delta_1(z_0) < \infty$. Let V be a neighborhood of z_0 such that (4) holds. Let $V'' \subset\subset V$, and $z_0 \in V''$. Then there exists $\epsilon > 0$ such that for all sufficiently small $\delta > 0$, there is a smooth plurisubharmonic function λ_δ in V'' such that $|\lambda_\delta| \leq 1$, and*

$$\begin{aligned} &\partial\overline{\partial}\lambda_\delta(L, \overline{L}) \gtrsim \delta^{-2\epsilon}|L|^2 \ , \\ &\partial\overline{\partial}\lambda_\delta(L, \overline{L}) \gtrsim |L\lambda_\delta|^2 \quad for \ \ z \in V'' \cap S(\delta). \end{aligned} \tag{5}$$

3. Proof of Theorem 3.

Let $V'' \subset\subset V$ be a neighborhood of $z_0 \in b\Omega_2$ such that Theorem 6 holds on V. In the proof of Theorem 3.5 in [7], the author also proved that

$$|D^\alpha \lambda_\delta| \le C_\alpha \delta^{-|\alpha|}, \tag{6}$$

for the plurisubharmonic weight functions λ_δ as in Theorem 6. Choose $V' \ni z_0$ so that $V' \subset\subset V''$. Now let us choose $z' \in V'$ and fix for a while. Let $D_R = \{z \in \mathbb{C}^n; |z| < R\}$ and let $\phi \in C_0^\infty(D_2 - D_{\frac{1}{4}})$ be a function that satisfies $\phi(z) = 1$ for $z \in D_1 - D_{\frac{1}{2}}$. Let N be a large integer to be chosen. For all $k > N$, set

$$\phi_k(z) = \phi(2^{k\epsilon} z), \tag{7}$$

where $\epsilon > 0$ is the number in Theorem 6. Also set

$$\phi_N(z) = \psi(2^{N\epsilon} z), \tag{8}$$

where $\psi \in C^\infty(\mathbb{C}^n)$, $\psi(z) = 1$ if $|z| \ge 2$, and $\psi(z) = 0$ if $|z| \le 1$. Therefore,

$$|D^\alpha \phi_k(z)| \le C_\alpha 2^{\epsilon k|\alpha|}, \tag{9}$$

for all $k \ge N$. Let a, t be numbers to be determined. Set $\psi_k(z', z) = \phi_k(z - z')$ and $\lambda_{2^{-k}a} = \Lambda_k$, where $\lambda_{2^{-k}a}$ is the plurisubharmonic function constructed as in Theorem 6. For $z \in V''$, define

$$E(z', z) = \sum_{k=N}^{\infty} 2^{-4k} \psi_k(z', z)(\Lambda_k(z) - 2). \tag{10}$$

By Theorem 6, Λ_k satisfies (5) and (6) in

$$V'' \cap S(2^{-k}a) = \{z \in V''; -2^{-k}a \le r(z) \le 2^{-k}a\},$$

provided N is sufficiently large. Suppose $z \in supp\psi_k$. Then $z \in supp\psi_j$ only if $|j - k| \le 3\epsilon^{-1}$ by (7) and (8). Therefore the sum in (10) is finite for each z, and $E(z', z)$ is a C^2-function by (6) and (10). We want to show that the Hessian of $E(z', z)$ satisfies a good lower bounds. Let $z \in supp\psi_k$ and $\psi_k(z) \ge \frac{1}{4}$. Then by (5) and (9),

$$
\begin{aligned}
&\partial\overline{\partial}(\psi_k(z', z)(\Lambda_k(z) - 2))(L, \overline{L}) \\
&= \partial\overline{\partial}\psi_k(z', z)(L, \overline{L})(\Lambda_k(z) - 2) + 2Re(L\psi_k)(\overline{L}\Lambda_k) + \psi_k \partial\overline{\partial}\Lambda_k(L, \overline{L}) \\
&\ge \partial\overline{\partial}\psi_k(z', z)(\Lambda_k(z) - 2) - 2C_\eta |L\psi_k|^2 - 2\eta|\overline{L}\Lambda_k|^2 + \frac{1}{4}\partial\overline{\partial}\Lambda_k(L, \overline{L}) \\
&\ge -C2^{2\epsilon k}|L|^2 - 2C'_\eta 2^{2\epsilon k}|L|^2 + \frac{1}{8}\partial\overline{\partial}\Lambda_k(L, \overline{L}) \\
&\ge (-C'2^{2\epsilon k} + \frac{1}{8}a^{-2\epsilon}2^{2\epsilon k})|L|^2 \\
&\gtrsim 2^{2\epsilon k}|L|^2,
\end{aligned}
$$

if a and η are sufficiently small. If we look at the proof of Theorem 3.5 in [7], we can see that $\partial\overline{\partial}\Lambda_k(z)(L,\overline{L}) \approx \partial\overline{\partial}\Lambda_j(z)(L,\overline{L})$ if $supp\psi_k \cap supp\psi_j \neq \emptyset$ (Remember $z \in supp\psi_j$ for a fixed finite number of j's independent of z'). Therefore we can get

$$\partial\overline{\partial}E(z',\tilde{z})(L,\overline{L}) \gtrsim 2^{-(4-2\epsilon)k}|L|^2, \tag{11}$$

if a and η are sufficiently small (depending on ϵ). Since $\lambda_{2^{-k}a}$ is defined on $S_{2^{-k}a}$ and since $E(z',z) \approx -2^{-4k}$, $\lambda_{2^{-k}a}$ and hence $E(z',z)$, is well defined on $S_{2^{-k}a}$ for all k sufficiently large (i.e., N is large enough). We now show how we push out the boundary of Ω near z'. By the estimate (6) one can get;

$$|D^\alpha E(z',z)| \lesssim a^{-|\alpha|}2^{(|\alpha|-4)k} \tag{12}$$

for $z \in supp\psi_k$. For all t sufficiently small, define

$$\rho_t(z',z) = r(z) + tE(z',z). \tag{13}$$

Then $\rho_t(z',\cdot)$ is a defining function of a hypersurface in V''. In fact since $\frac{\partial r}{\partial x_n} \approx 1$ ($z_n = x_n + iy_n$), it follows that for all small t, $\frac{\partial \rho_t}{\partial x_n}(z',z) \approx 1$. Hence for any $z \in V'' \cap b\Omega$, there exists only one point $\tilde{z}$ on the segment through z obtained by varying x_n, which satisfies $\rho_t(z',\tilde{z}) = 0$, for all $t \leq t_0$, provided that t_0 is sufficiently small. Notice that this t_0 is independent of $z' \in V'$. Set $S_{t,z'} = \{z; \rho_t(z',z) = 0\}$. Since $E(z',z') = 0$ and $E(z',z) < 0$ for $z \in V'' - \{z'\}$, the properties (i), (ii), (iii), (iv) and (vi) clearly holds. To prove (v), we claim that the hypersurface $S_{t,z'}$ is pseudoconvex at each point $\tilde{z} \in S_{t,z'}$, provided z is sufficiently close to z', say $|z - z'| < d$ where d is independent of z'. Suppose $L''\rho_t(z',\tilde{z}) = 0$, and $|L''| = 1$. Then L'' can be written as,

$$L'' = t_1L_1 + ... + t_{n-1}L_{n-1} + eL_n = T + eL_n,$$

where $L_1, ...L_{n-1}, L_n$ are local frames defined on V'' such that $L_jr = 0$, $j = 1, 2, ..., n-1$, and $L_nr > 0$ on V''. Let $\tilde{z} \in supp\psi_k$ and $\psi_k(z',\tilde{z}) \geq \frac{1}{4}$. Then,

$$\begin{aligned} L''\rho_t(z',\tilde{z}) &= L''r(\tilde{z}) + tL''E(z',\tilde{z}) \\ &= Tr(\tilde{z}) + eL_nr(\tilde{z}) + t(TE(z',\tilde{z}) + eL_nE(z',\tilde{z})) \\ &= e(L_nr(\tilde{z}) + tL_nE(z',\tilde{z})) + tTE(z',\tilde{z}) = 0. \end{aligned}$$

Since $|tL_nE(z',\tilde{z})| << 1$, we have $|e| \lesssim t|TE(z',\tilde{z})| \lesssim t << \frac{1}{2}$ if t is sufficiently small. Again if $|j-k| \leq 3\epsilon^{-1}$, then we can get

$$\partial\overline{\partial}E(z',\tilde{z})(T,\overline{T}) \gtrsim 2^{-4k}\partial\overline{\partial}\Lambda_j(\tilde{z})(T,\overline{T}) \gtrsim 2^{-4k}|T\Lambda_j(\tilde{z})|^2 \tag{14}$$

and

$$\partial\overline{\partial}E(z',\tilde{z})(T,\overline{T}) \gtrsim 2^{-4k}\partial\overline{\partial}\Lambda_k(T,\overline{T}) \gtrsim 2^{-(4-2\epsilon)k}|T|^2 \gtrsim 2^{-(4-2\epsilon)k}. \tag{15}$$

This implies that

$$\begin{aligned} &|T\Lambda_j(\tilde{z})| \lesssim 2^{2k}(\partial\overline{\partial}E(z',\tilde{z})(T,\overline{T}))^{\frac{1}{2}}, \quad \text{and} \\ &|T\psi_j(z',\tilde{z})| \lesssim 2^{k\epsilon} \lesssim 2^{2k}(\partial\overline{\partial}E(z',\tilde{z})(T,\overline{T}))^{\frac{1}{2}}. \end{aligned}$$

Thus we obtain that

$$|e| \lesssim t|TE(z',\tilde{z})| \lesssim t2^{-2k}(\partial\overline{\partial}E(z',\tilde{z})(T,\overline{T}))^{\frac{1}{2}}. \tag{16}$$

If we combine (12), (14), (15) and (16) we can get;

$$\begin{aligned}
\partial\overline{\partial}\rho_t(z',\tilde{z})(L'',\overline{L}'') &= \partial\overline{\partial}r(\tilde{z})(L'',\overline{L}'') + t\partial\overline{\partial}E(z',\tilde{z})(L'',\overline{L}'') \\
&= \partial\overline{\partial}r(\tilde{z})(T,\overline{T}) + t\partial\overline{\partial}E(z',\tilde{z})(T,\overline{T}) + \mathcal{O}(e) \\
&\geq t(\partial\overline{\partial}E(z',\tilde{z})(T,\overline{T}))^{\frac{1}{2}}((\partial\overline{\partial}E(z',\tilde{z})(T,\overline{T}))^{\frac{1}{2}} - C2^{-2k}) \\
&\gtrsim t(\partial\overline{\partial}E(z',\tilde{z})(T,\overline{T}))^{\frac{1}{2}}(2^{(-2+\epsilon)k} - C2^{-2k}) \\
&\geq t2^{-2k}(\partial\overline{\partial}E(z',\tilde{z})(T,\overline{T}))^{\frac{1}{2}}(2^{\epsilon k} - C) \\
&\gtrsim t2^{(-4+\epsilon)k}|T|^2 \\
&\gtrsim t2^{(-4+\epsilon)k}|L''|^2,
\end{aligned}$$

if k is sufficiently large (i.e., N is large enough). Here we have used the fact that $\partial\overline{\partial}r(\tilde{z})(T,\overline{T}) \geq 0$ and $|T| \geq \frac{1}{2}$. Also, we may assume that N can be chosen independently to $z' \in V' \cap b\Omega_2$. This proves (v).

To prove (vii), let $z \in supp\psi_k$. Then $d(z',z) \approx 2^{-2\epsilon k}$, and $\rho_t(z',z) \approx -2^{-4k}$. Therefore (vii) holds for all $z \in V'' \cap \{z : r(z) \leq 0\}$ with $K = 2\epsilon^{-1}$, and this proves Theorem 3. □

Remark. *In (7) if we replace $\phi_k(z)$ by $\phi_k(z) = \phi(kz)$ and $\Lambda_k = \lambda_{2^{-k}a}$ by $\Lambda_k = \lambda_{k^{-2\epsilon^{-1}}}$, then the hypersurfaces $\{z : \rho_t(z,z') = 0\}$ in Theorem 3 will be C^∞. Therefore we have a smooth bumping theorem. But in this case, the property (vii) of Theorem 3 will not be true.*

4. Boundary behavior of the Kobayashi metric.

We now want to use the result of Theorem 3 to prove a boundary behavior of the Kobayashi metric, K_Ω, near a point z_0 of finite type, i.e., K_Ω tends to infinite near z_0 at least as $r^{-\epsilon'}$ does for a defining function r of Ω near z_0 and some $\epsilon' > 0$. We adopt Diederich and Fornaess' method [10] to estimate the Kobayashi metric near $b\Omega_2$.

Theorem 7. *([10], Theorem 3) Let r be a real-valued C^2-function on a neighborhood $\widetilde{V}'' \subset \mathbb{C}^n$ of 0 with the following properties:*

1. *$r(0) = 0$*
2. *$dr \neq 0$ everywhere on $\widetilde{V}''$.*
3. *The hypersurface $S = \{z \in \widetilde{V}'' : r(z) = 0\}$ is pseudoconvex from the side $r < 0$.*

Then, for every η, $0 < \eta < 1$, there exists an open neighborhood $\widetilde{V}' \subset \widetilde{V}''$ of 0, a strictly plurisubharmonic function ρ on $U = \{z \in \widetilde{V}' : r(z) < 0\}$ and a constant $C \geq 1$ such that $-C|r|^\eta < \rho < -\frac{1}{C}|r|^\eta$ on U. Furthermore, the data C and $\widetilde{V}'$ can be chosen independently of small C^2-perturbations of r on $\widetilde{V}''$ satisfying conditions (1), (2), (3) from above.

Let us now prove Theorem 1. Let $z_0 \in b\Omega$ and $V'' \supset\supset V' \ni z_0$ be the neighborhoods as in Theorem 3. Choose also $t_0 > 0$ sufficiently small so that Theorem

3 holds on V'', and set $\rho(z', z) = \rho_{t_0}(z', z)$. In a first step we apply Theorem 7 to the family of hypersurfaces

$$S_{z'} = \{z \in V''; \rho(z', z) = 0\}$$

for $z' \in b\Omega \cap V'$. According to Theorem 7, we can choose an open neighborhood $\widetilde{V}' \subset \widetilde{V}''$ of z_0 in the z'-variable, a constant $C > 0$ such that for each $z' \in b\Omega \cap \widetilde{V}'$, there is a strictly plurisubharmonic function $\phi_{z'}(z)$ on $\widetilde{V}'' \cap \{\rho(z', z) < 0\}$ satisfying

$$-C|\rho(z', z)|^\eta < \phi_{z'}(z) < -\frac{1}{C}|\rho(z', z)|^\eta \tag{17}$$

for all $z \in \widetilde{V}''$ and for all $z' \in b\Omega \cap \widetilde{V}'$. If we shrink V' and V'', we may assume that $V' = \widetilde{V}'$ and $V'' = \widetilde{V}''$. Combining (17) and the property (vii) of Theorem 3, one can get the important estimate

$$\frac{1}{C}(d(z', z))^{2\epsilon^{-1}\eta}(z', z) \leq |\phi_{z'}(z)| \tag{18}$$

for all $z \in \overline{V}'' \cap \overline{\Omega}$, $z' \in b\Omega \cap V'$. The estimates (17), (18) and (vii) of Theorem 3 are the essential estimates necessary to prove the boundary behavior of Kobayashi metric on Ω in [10]. If we follow Diederich and Fornaess procedure from this point on, we can get, for all $z \in V'' \cap \Omega$ and for all $X \in \mathbb{C}^n$,

$$K_\Omega(z; X) \geq c|X||r(z)|^{-\epsilon/4} \tag{19}$$

with a constant $c > 0$ independent of z and X.

Remarks

(1) By the Remark in [10], $\epsilon'' > 0$ in Corollary 2 can be chosen arbitrary close to ϵ', i.e., $\epsilon'' = \epsilon' - \mu$ for arbitrary small $\mu > 0$.
(2) Using the holomorphic coordinate functions up to the boundary [4], and using the definition of finite type for the manifolds [6,7], Theorem 1 can be proved if $\Omega_1 \subset\subset M_1$ and $\Omega_2 \subset\subset M_2$ for some Stein manifolds M_1, M_2.

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