HOLOMORPHIC APPROXIMATION ON COMPACT PSEUDOCONVEX COMPLEX MANIFOLDS

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ABSTRACT. Let $\overline{M}$ be a smoothly bounded compact pseudoconvex complex manifold of finite type in the sense of D'Angelo such that the complex structure of M extends smoothly up to bM . Let m be an arbitrary nonnegative integer. Let f be a function in $H(M) \cap H_m(M)$, where $H_m(M)$ is the Sobolev space of order m . Then f can be approximated by holomorphic functions on $\overline{M}$ in the Sobolev space $H_m(M)$. Also, we get a holomorphic approximation theorem near a boundary point of finite type.

1. Introduction and statement of the results.

In [14], Mergelyan proved that if $K \subset \mathbb{C}$ is compact and $\mathbb{C} - K$ has just one connected component, then every $f \in C(K)$ which is holomorphic in the interior of K can be approximated uniformly on K by functions holomorphic in some neighborhood of K (and hence by polynomials). One cannot obtain the Mergelyan approximation property in several complex variables only with the simple topological hypothesis. Surprisingly, Diederich and Fornaess [8] constructed a smoothly bounded pseudoconvex domain D in $\mathbb{C}^2$ and a function $f \in A^\infty(D) = H(D) \cap C^\infty(\overline{D})$ which can not be uniformly approximated by functions in $H(\overline{D})$. However, there is a classical result that proves the Mergelyan approximation property:

Theorem (Kerzman [12], Henkin [11], Lieb [13]). *Let $D \Subset \mathbb{C}^n$ be a smoothly bounded strongly pseudoconvex domain. Then any function continuous on $\overline{D}$ and holomorphic on D can be approximated uniformly on $\overline{D}$ by functions holomorphic on $\overline{D}$.*

In the weakly pseudoconvex case, Fornaess and Nagel [9] got the same result when $D \Subset \mathbb{C}^2$ is a pseudoconvex domain with real analytic boundary. Other generalizations can be found in [1, 15]. The purpose of this paper is to get the Mergelyan approximation property on compact pseudoconvex complex manifolds of finite type in the sense of D'Angelo [7]. However, instead of using the uniform convergence topology, we use the topology given in Sobolev spaces.

Let $\overline{M}$ be a smoothly bounded compact manifold and $f \in L^2(M)$. Let $\zeta_j, j = 1, 2, \dots, N$, be a partition of unity subordinated to a covering of $\overline{M}$. We define

$$\|f\|_{m,M}^2 = \sum_{j=1}^N \sum_{|\alpha| \leq m} \|D^\alpha(\zeta_j f)\|^2$$

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where D^α refers to $\partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_{2n}^{\alpha_{2n}}$ for the coordinate neighborhood of $\text{supp } \zeta_j$. Then we let the m -th order Sobolev space, $H_m(M)$, be the space of all $f \in L^2(M)$ such that $\|f\|_{m,M} < \infty$.

In [4], Catlin proved the existence of local holomorphic coordinate functions up to the boundary in case $\overline{M}$ is a pseudoconvex complex manifold. In the sense of D'Angelo, finite type depends on the finite order of Taylor polynomial of the defining function for $\overline{M}$. Therefore, it is reasonable to define a finite type for the manifolds in conjunction with holomorphic coordinates.

Definition 1.1. Let $\overline{M}$ be a compact pseudoconvex complex manifold and $z_0 \in bM$. Then we have a neighborhood U of z_0 and holomorphic coordinate functions $f = (f_1, \dots, f_n)$ defined on U . In this case, we define; $\text{Type of } z_0 = \text{Type of } f(z_0)$ on $f(bM)$ in $\mathbb{C}^n$.

Since the type condition is an invariant property, it is clear that the above definition of type does not depend on coordinate functions. For compact pseudoconvex complex manifold $\overline{M}$ of finite type ($\dim_{\mathbb{C}} M = n \geq 2$), the second author [6] showed that $\overline{M}$ can be holomorphically embedded into a larger complex manifold Ω of the same complex dimension. In virtue of this result, we can now state our main theorems as follows:

Theorem 1.1. *Let $\overline{M}$ be a compact pseudoconvex complex manifold with smooth boundary. Assume also that bM is finite type and $\dim_{\mathbb{C}} M \geq 2$. Let f be a function in $H(M) \cap H_m(M)$, where m is a nonnegative integer. Then there exists a sequence of functions f_n in $H(\overline{M})$ such that*

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{m,M} = 0.$$

Among the many uses of the holomorphic approximation theorem, we mention the construction of peak functions [2] and the stability of the Bergman kernel function [10]. A result on the interior stability of the Bergman kernel function is given in Section 3.

Also, we get a holomorphic approximation theorem near a boundary point of finite type.

Theorem 1.2. *Let $\overline{M}$ and r be as in Theorem 1.1 without the assumption of finite type. Suppose there is a C^∞ plurisubharmonic function φ on $\overline{M}$ which is strongly plurisubharmonic on a neighborhood of bM . Let z_0 be a point of finite type in the boundary of M . Let f be a function in $H(M) \cap H_m(M)$, where m is a nonnegative integer. Then there exist a neighborhood V of z_0 and a sequence of functions f_n in $A^\infty(M)$ which extend holomorphically on $V \cap b\Omega$ such that*

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{m,M} = 0.$$

Now let D be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$. Then $\varphi(z) = |z|^2$ is a C^∞ strongly plurisubharmonic function on $\overline{D}$. Therefore, we obtain the following immediate consequence of Theorem 1.2.

Corollary 1.3. *Let z_0 be a point of finite type in the boundary of D . Let f be a function in $H(D) \cap H_m(D)$, where m is a nonnegative integer. Then there exist*

a neighborhood V of z_0 and a sequence of functions f_n in $A^\infty(D)$ which extend holomorphically on $V \cap b\Omega$ such that

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{m,M} = 0.$$

2. A stability theorem for the $\bar{\partial}$ -Neumann problem.

Let $\bar{M}$ be as in Theorem 1.1 and let $M' \supset \bar{M}$ be a C^∞ manifold. Assume that there exist a neighborhood V of bM and a real-valued function $r \in C^\infty(V)$ such that $\nabla r \neq 0$ on bM and $r(z) = 0$ if and only if $z \in bM$. The sign of r is chosen so that $r < 0$ on M and $r > 0$ outside of $\bar{M}$. In a local coordinate chart U of a complex manifold $\bar{M}$, any C^∞ vector field can be written as $L = \sum_{j=1}^{2n} a_j \frac{\partial}{\partial x_j}$ where $a_j \in C^\infty(U)$ and $x_j, j = 1, 2, \dots, 2n$, are local (real-valued) coordinate functions. Therefore we can define a C^∞ -metric for a C^∞ vector field L by

$$\|L\|_{C^\infty(U)} = \max_j \{\|a_j\|_{C^\infty(U)}\}$$

where $\|a_j\|_{C^\infty(U)}$ is a C^∞ -metric for C^∞ functions.

Definition 2.1. Let $I \subset \mathbb{R}^l$ be a domain containing 0. Then the family of complex manifolds $\{M_\tau\}_{\tau \in I}$, $M_\tau \subset M'$, with smooth defining functions r_τ for $\tau \in I$, is said to be a continuous family of diffeomorphic complex manifolds with diffeomorphisms $d_\tau : M_\tau \rightarrow M_0$, if

- (1) $d_0 : M_0 \rightarrow M_0$ is the identity,
- (2) the complex structures on M_τ are C^∞ close to the complex structure on M_0 as $\tau \rightarrow 0$,
- (3) r_τ and all of its derivatives are continuous functions with respect to τ ,
- (4) the diffeomorphisms d_τ are continuous functions of τ .

Let $\|\cdot\|_{m,t}$ be the m -th order weighted Sobolev norm with weight $e^{-t\varphi}$ and let $H^{0,1}(M_\tau)$ be the $\bar{\partial}$ -cohomology group on M_τ . Here φ is a smooth function on $\bar{M}_\tau$ for all $\tau \in I$.

Theorem 2.2 ([5, Theorem 3.4]). Let $\{\bar{M}_\tau\}_{\tau \in I}$ be a continuous family of smoothly bounded diffeomorphic compact pseudoconvex complex manifolds in M' . Suppose there is a C^∞ plurisubharmonic function φ defined on a neighborhood of $\cup_\tau \bar{M}_\tau$ which is strongly plurisubharmonic on a neighborhood of bM_τ for all $\tau \in I$. Then there exists a constant $C_{m,t}$ which does not depend on τ , and there is a neighborhood I' of $\tau = 0$, such that

$$\|f^\tau\|_{m,t} \leq C_{m,t} \|\square_\tau f^\tau\|_{m,t}$$

for all $f^\tau \in D_{\square_\tau} \cap C^\infty(\bar{M}_\tau)$ with $f^\tau \perp H^{0,1}(M_\tau)$, $\tau \in I'$. Here $\square_\tau$ denotes the usual complex Laplacian on M_τ and $D_{\square_\tau}$ denotes the domain of $\square_\tau$.

3. A proof of the approximation theorems.

In [6], the second author constructed a continuous family of diffeomorphic strongly pseudoconvex compact complex manifolds which shrinks nicely to a compact pseudoconvex complex manifold of finite type. Such a well-behaved continuous family and a stability theorem for the $\bar{\partial}$ -Neumann problem are ingredients to prove our approximation theorem on a compact pseudoconvex complex manifold of finite type. Also, we make use of the following density lemma.

Lemma 3.1 ([3, Proposition 3.1.4]). *Let $\overline{M}$ be a smoothly bounded compact pseudoconvex complex manifold. Suppose there is a C^∞ plurisubharmonic function φ on $\overline{M}$ which is strongly plurisubharmonic on a neighborhood of bM . Let f be a function in $H(M) \cap H_m(M)$, where m is a nonnegative integer. Then for any $\epsilon > 0$, there exists $g \in A^\infty(M)$ with $\|g - f\|_{m,M} < \epsilon$.*

Proof of Theorem 1.1. By using the bumping theorem [6, Theorem 2.3], Cho constructed a smoothly bounded pseudoconvex complex manifold Ω in M' so that $\overline{M}$ can be holomorphically embedded into Ω . Also in [6, Lemma 4.2], he proved that there exists $\epsilon > 0$ such that for all sufficiently small $\delta > 0$, there is a smooth plurisubharmonic function $\varphi \in C^\infty(\overline{M})$ such that $|\varphi| \leq 1$, and for all smooth vector field L ,

$$(3.1) \quad \begin{aligned} \partial\bar{\partial}\varphi(L, \overline{L}) &\gtrsim \delta^{-2\epsilon}|L|^2, \\ \partial\bar{\partial}\varphi(L, \overline{L}) &\gtrsim |L\varphi|^2 \text{ for } z \in S(\delta), \end{aligned}$$

where $S(\delta) = \{z \in M' : -\delta < r(z) < \delta\}$. Thus φ is strongly plurisubharmonic on $S(\delta_0)$ for some $\delta_0 > 0$. Define $r_\delta(z) = r(z) + \delta(\varphi - 2)$ and set $\Omega_\delta = \{z \in M' : r_\delta(z) < 0\}$. Since $|\varphi| \leq 1$, there exists $\delta_0 > 0$ such that $\Omega_\delta \subset \overline{\Omega}$ and φ is defined near $b\Omega_\delta$ for all $\delta \leq \delta_0$. Using (3.1), we can prove that each Ω_δ is a strongly pseudoconvex complex manifold for all $0 < \delta \leq \delta_0$ provided δ_0 is sufficiently small (See [6, Proof of Theorem 1]). Thus $\{\overline{\Omega}_\delta\}_{0 < \delta \leq \delta_0}$ is a continuous family of diffeomorphic strongly pseudoconvex compact complex manifolds such that $\overline{\Omega}_0 = \overline{M} \Subset \Omega_\delta$ for all $0 < \delta \leq \delta_0$.

By Lemma 3.1, $A^\infty(M)$ is dense in $H(M) \cap H_m(M)$. Thus it is enough to prove our theorem in the case of functions f in $A^\infty(M)$. Let $f \in A^\infty(M)$ and define $f_\delta = P_t^\delta(d_\delta^*f)$, where $d_\delta : \overline{\Omega}_\delta \rightarrow \overline{\Omega}_0$ are diffeomorphisms and $P_t^\delta : L^2(\Omega_\delta, t\varphi) \rightarrow L^2(\Omega_\delta, t\varphi) \cap H(\Omega_\delta)$ are Bergman projections with respect to the weight $e^{-t\varphi}$. Then f_δ is holomorphic on Ω_δ and

$$f_\delta = d_\delta^*f - \bar{\partial}_\delta^* N_t^\delta \bar{\partial}_\delta d_\delta^*f,$$

where N_t^δ is the Neumann operator on Ω_δ with respect to the weight $e^{-t\varphi}$. By Theorem 2.2, for any nonnegative integer $m \geq 0$,

$$\begin{aligned} \|\bar{\partial}_\delta^* N_t^\delta \bar{\partial}_\delta d_\delta^*f\|_{t,m} &\lesssim \|N_t^\delta \bar{\partial}_\delta d_\delta^*f\|_{t,m+1} \\ &\lesssim \|\bar{\partial}_\delta d_\delta^*f\|_{t,m+1} \end{aligned}$$

uniformly in δ near 0. Since the complex structures on $\overline{\Omega}_\delta$ converge to the complex structure on $\overline{\Omega}_0 = \overline{M}$ in C^∞ -topology, we can get $\bar{\partial}_\delta d_\delta^*f \rightarrow \bar{\partial}f = 0$, also in C^∞ -topology as $\delta \rightarrow 0$. So $\|\bar{\partial}_\delta^* N_t^\delta \bar{\partial}_\delta d_\delta^*f\|_{t,m}$ converges to zero as $\delta \rightarrow 0$. Since the diffeomorphisms d_δ are continuous functions of δ , $d_\delta^*f \rightarrow f$ in C^∞ -topology on $\overline{M}$. Notice that the weighted Sobolev space is the usual Sobolev space with a different, but comparable norm. Thus

$$\lim_{\delta \rightarrow 0} \|f - f_\delta\|_{m,M} = 0.$$

This proves the theorem 1.1. $\square$

By extremal properties of the Bergman kernel function, the holomorphic approximation theorem implies the interior stability of the Bergman kernel function (See [10]). Thus we obtain the following theorem as a corollary of Theorem 1.1:

Theorem 3.2. *Let D_0 be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ of finite type in the sense of D'Angelo. Let C be a compact subset of D_0 . Given $\epsilon > 0$, there exists a domain D containing $\overline{D}_0$ for which*

$$\sup_{z, w \in C \cap D} |K_D(z, w) - K_{D_0}(z, w)| < \epsilon.$$

Proof of Theorem 1.2. In [6, Theorem 2.3], the second author proved that there exist a neighborhood V of z_0 and a smooth 1-parameter family of compact pseudoconvex complex manifolds $\overline{M}_\delta, 0 \leq \delta < \delta_0$, each defined by $\overline{M}_\delta = \{z \in M' : r_\delta(z) \leq 0\}$ and satisfying the following properties:

- (1) $\overline{M}_0 = \overline{M}$.
- (2) $\overline{M}_{\delta_1} \subset \overline{M}_{\delta_2}$ if $\delta_1 < \delta_2$.
- (3) $\overline{M}_\delta - V = \overline{M}_0 - V$.
- (4) There exist a family of diffeomorphisms $d_\delta : \overline{M}_\delta \rightarrow \overline{M}_0$ such that $d_0 = Id$ and $d_\delta = Id$ on $\overline{M}_\delta - V$.
- (5) r_δ and all of its derivatives are continuous functions with respect to δ .

For any vector field L near bM_δ , it follows that

$$\partial \bar{\partial} d_\delta^* \varphi(L, \bar{L}) = \partial \bar{\partial} \varphi((d_\delta)_* L, (d_\delta)_* \bar{L}) \geq c |(d_\delta)_* L|^2 > \frac{c}{2} |L|^2$$

provided δ is sufficiently small. Thus we may assume that φ is strongly plurisubharmonic in a neighborhood of bM_δ for all $0 \leq \delta < \delta_0$.

First we assume $f \in A^\infty(M)$. Define $f_\delta = P_t^\delta(d_\delta^* f)$. Then as in the proof of Theorem 1.1, we can prove that $f_\delta \in H(M_\delta) \cap H_m(M_\delta)$ and that for any $\epsilon > 0$, there exists $\delta' < \delta_0$ such that

$$\|f_\delta - f\|_{m, M} < \frac{\epsilon}{2} \quad \text{for all } \delta < \delta'.$$

Since $A^\infty(M_\delta)$ is dense in $H(M_\delta) \cap H_m(M_\delta)$, there exists $f'_\delta \in A^\infty(M_\delta)$ such that $\|f_\delta - f'_\delta\|_{m, M_\delta} < \frac{\epsilon}{2}$ for each $\delta < \delta'$. Thus $f'_\delta \in H(M) \cap C^\infty(\overline{M})$ extends holomorphically across z_0 and

$$\|f - f'_\delta\|_{m, M} < \epsilon \quad \text{for all } \delta < \delta'.$$

Since $A^\infty(M)$ is dense in $H(M) \cap H_m(M)$, we also get the theorem 1.2. $\square$

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