HOLOMORPHIC SECTIONAL CURVATURE
OF THE BERGMAN METRIC IN $\mathbb{C}^n$.

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Abstract. Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth defining function $r$ and let $z_0 \in b\Omega$ be a point of finite type. We also assume that the Levi form $\partial \overline{\partial} r(z)$ of $b\Omega$ has $(n-2)$-positive eigenvalues at $z_0$. Then we prove that all the holomorphic sectional curvatures of the Bergman metric of $\Omega$ are bounded below by a negative constant near $z_0$.

1. Introduction.

Many questions about the complex function theory of a domain in $\mathbb{C}^n$ can be explored by examining the curvature of the domain in an appropriate Hermitian metric. These include the curvature conditions to characterize domains of holomorphy [9], extension of the classical Schwarz lemma [7, 10]. The abstract nature of the metrics makes it difficult to obtain informations about the curvature tensor except in some special cases. It has been well known that for any bounded domain in $\mathbb{C}^n$, the holomorphic sectional curvature of the Bergman metric is less than or equal to 2. Using Fefferman’s asymptotic expansion of the Bergman kernel, Klembeck [6] showed that for a smooth bounded strongly pseudoconvex domain in $\mathbb{C}^n$, the holomorphic sectional curvatures of the Bergman metric approach $-\frac{4}{n+1}$, that of the ball, near the boundary. For weakly pseudoconvex domains, however, much less is known for the lower bounds of the holomorphic sectional curvatures. In [8], J. McNeal showed that the holomorphic sectional curvatures of the Bergman metric of pseudoconvex domains in $\mathbb{C}^2$ are bounded below by a negative constant near the boundary point of finite type. In this paper, we estimate the sectional curvatures of the Bergman metric for a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ near the boundary point of finite type. The result is

Theorem. Let $\Omega \subset \subset \mathbb{C}^n$ be a pseudoconvex domain with smooth boundary. If $z_0 \in b\Omega$ is a point of finite type and the Levi-form of $b\Omega$ has $(n-2)$-positive eigenvalues at $z_0$, then there is a neighborhood $U$ of $z_0$ such that all the holomorphic sectional curvatures of the Bergman metric of $\Omega$ are bounded below by a negative constant in $U$.

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2. preliminaries.

Let $\Omega$ be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth defining function $r$, i.e., $dr \neq 0$ on $b\Omega$ and $\Omega = \{ z \in \mathbb{C}^n : r(z) < 0 \}$. We denote by $A(\Omega)$ the set of holomorphic functions on $\Omega$. Let $X$ be a holomorphic tangent vector at a point $z \in \Omega$. Then the Bergman kernel $K(z, z)$ and the Bergman metric $B(z; X)$ are defined

$$K_\Omega(z, \Bar{z}) = \sup \{ |f(z)|^2 : f \in A(\Omega), \|f\|_{L^2(\Omega)} \leq 1 \}$$
$$B_\Omega(z; X) = \sup \{ |Xf| : f \in A(\Omega), f(z) = 0, \|f\|_{L^2(\Omega)} \leq 1 \}$$

Let $z_0 \in b\Omega$ be a point of finite type $m$ in the sense of D'Angelo [5] and assume that the Levi form $\partial \Bar{\partial} r(z)$ of $b\Omega$ has $(n-2)$-positive eigenvalues at $z_0$. Note that the type $m$ at $z_0$ is an even integer in this case. We may assume that there are coordinate functions $z_1, \ldots, z_n$ such that $|\frac{\partial r}{\partial z_j}(z)| \geq c > 0$ for all $z$ in a neighborhood $U$ of $z_0$. It will be convenient to express the necessary estimates of the above quantities in terms of the special coordinate systems used in [2,3,4].

**Proposition 2.1 ([2, Proposition 2.2]).** For each $z' \in U$ and positive even integer $m$, there is a biholomorphism $\Phi_{z'} : \mathbb{C}^n \rightarrow \mathbb{C}^n$, $\Phi_{z'}^{-1}(z') = 0$, $\Phi_{z'}^{-1}(z) = (\zeta_1, \ldots, \zeta_n)$ such that

$$r(\Phi_{z'}(\zeta)) = r(z') + Re \zeta_1 + \sum_{a=2}^{n-1} \sum_{j+k \leq \frac{m}{2}} Re \left( b_{j,k}^{a}(z') \zeta_n^j \zeta_1^k \right)$$
$$+ \sum_{\substack{j+k \leq m \cr j,k > 0}} a_{j,k}(z') \zeta_n^j \zeta_1^k + \sum_{\alpha=2}^{n-1} |\zeta_\alpha|^2$$
$$+ \mathcal{O}(|\zeta_1||\zeta| + |\zeta''||\zeta| + |\zeta'''||\zeta_n|^{\frac{m}{2}} + |\zeta_n|^{m+1}).$$

(1)

**Remark 2.2.** The coordinate changes in the proof of Proposition 2.1 are canonical in the sense that the map $\Phi_{z'}$ is defined uniquely and it normalizes the defining function to the form (1). Note also that

$$|J\Phi_{z'}(\zeta)| \geq c' > 0, \quad \zeta \in \Phi_{z'}^{-1}(U),$$

for some $c' > 0$, independent of $z'$.

Set $\Phi_{z'}^{-1}(U) = V$, and $\Phi_{z'}^{-1}(\Omega) = \Omega_{z'}$, and set $\rho(\zeta) = r \circ \Phi_{z'}(\zeta)$. Define

$$A_l(z') = \max \{ |a_{j,k}(z')| : j + k = l \}, \quad 2 \leq l \leq m,$$
$$B_{l'}(z') = \max \{ |b_{j,k}^{a}(z')| : j + k = l' \}, \quad 2 \leq \alpha \leq n-1, \quad 2 \leq l' \leq \frac{m}{2}.$$

For each $\delta > 0$, we define $\tau(z', \delta)$ as follows

$$\tau(z', \delta) = \min \{ (\delta/A_l(z'))^{\frac{1}{2}}, (\delta^{\frac{1}{2}}/B_{l'}(z'))^{\frac{1}{2}} : 2 \leq l \leq m, \quad 2 \leq l' \leq \frac{m}{2} \}.$$
In [2], the first author proved that \((\delta^{\frac{1}{2}}/B_U(z'))\frac{1}{\delta} >> \tau\) and hence

\[
\tau(z', \delta) = \min\{(\delta/A_\ell(z'))^{\frac{1}{2}} : 2 \leq \ell \leq m\}.
\] (2)

As \(z_0\) is of finite type \(m\), it follows that \(A_m(z_0) \neq 0\) and hence we may assume that \(A_m(z') \geq \epsilon > 0\) for all \(z' \in U\) if \(U\) is sufficiently small. This gives the inequality,

\[
\delta^{\frac{1}{2}} \leq \tau(z', \delta) \lesssim \delta^\frac{1}{m}, \quad z' \in U.
\]

Now set \(\tau_1 = \delta, \tau_2 = \ldots = \tau_{n-1} = \delta^{\frac{1}{2}}, \tau_n = \tau(z', \delta) = \tau\) and define

\[
R_\delta(z') = \{\zeta \in \mathbb{C}^n; |\zeta_k| < \tau_k, k = 1, 2, \ldots, n\}, \quad \text{and} \quad Q_\delta(z') = \{\Phi(z'); \zeta \in R_\delta(z')\}.
\]

The importance of the quantity \(\tau_n = \tau(z', \delta)\) rests on the fact that \(R_\delta(z')\) is essentially the largest polydisc about the origin on which \(\rho\) changes by no more than \(\delta\). One can refer [2] for details.

We now recall the estimates from [2,3,4] for the Bergman kernel function and Bergman metric for \(\Omega_{z'}\) along the normal ray at the origin. The theorems in [2,3], show there exists a constant \(b\), independent of \(z' \in U \cap b\Omega\) and small \(\delta \in (0, \delta_0)\), so that for \(\zeta = \left(-\frac{b\delta}{2}, 0, \ldots, 0\right)\)

\[
K_{\Omega_{z'}}(\zeta, \zeta) \approx \delta^{-n} \tau(z', \delta)^{-2},
\] (3)

and

\[
B_{\Omega_{z'}}(\zeta; Y) \approx |c_1|\delta^{-1} + |c_2|\delta^{-\frac{1}{2}} + \ldots + |c_{n-1}|\delta^{-\frac{1}{2}} + |c_n|\tau(z', \delta)
\] (4)

if \(Y = c_1 \frac{\partial}{\partial \zeta_1} + \ldots + c_n \frac{\partial}{\partial \zeta_n}\). In [4], the first author got an estimate

\[
|D_\zeta^\alpha D_\zeta^\beta K_{\Omega_{z'}}(\zeta, \zeta)| \leq C\delta^{-n-\gamma} \tau(z', \delta)^{-2-\alpha_n-\beta_n},
\] (5)

where \(\gamma = \alpha_1 + \beta_1 + \frac{1}{2} (\alpha_2 + \beta_2 + \ldots + \alpha_{n-1} + \beta_{n-1})\), for any \(n\)-indices \(\alpha, \beta\). Note that the constants in the above estimates are independent of \(z'\) and \(\delta\).

### 3. Curvature estimates.

For the time being, \(z'\) will be fixed, and we will denote \(K_{\Omega_{z'}}\), by \(K\) and derivatives of \(K_{\Omega_{z'}}\) with subscripts, e.g. \(\frac{\partial^2}{\partial z_i \partial \zeta_j} K_{\Omega_{z'}} = K_{ij}\). Set, for \(1 \leq i, j \leq n\) and \(\zeta \in V\),

\[
g_{ij}(\zeta) = \frac{\partial^2}{\partial \zeta_i \partial \zeta_j} \log K(\zeta, \zeta).
\]

Then, if \(Y = c_1 \frac{\partial}{\partial \zeta_1} + \ldots + c_n \frac{\partial}{\partial \zeta_n}\), it is well known [1] that

\[
B_{\Omega_{z'}}(\zeta; Y) = \left(\sum_{i,j=1}^{n} g_{i,j}(\zeta) c_i c_j\right)^{\frac{1}{2}}.
\] (6)
If we compare coefficients in (4) and (6), we have

\[ |g_{ii}(\zeta)| \approx \tau_i^{-2}, \quad |g_{ij}(\zeta)| \lesssim \tau_i^{-1} \tau_j^{-1}. \]

Set \( G = (g_{ij})_{1 \leq i,j \leq n} \) and let \( P = (P_{jk}) \) be a unitary matrix such that

\[ P^* G P = D \]

where \( D \) is a diagonal matrix whose entries are positive eigenvalues of \( G \). For \( x \in \mathbb{C}^n \) with \( |x| = 1 \), set \( c = Px \). Then by (4),

\[ c^* G^* c = x^* D x = \lambda_1 |x_1|^2 + \ldots + \lambda_n |x_n|^2 \]
\[ \approx |c_1|^2 \tau_1^{-2} + \ldots + |c_n|^2 \tau_n^{-2}. \]

Set \( x = x^k = (0, \ldots, 1, 0, \ldots, 0) \), where 1 is in the \( k \)-th place. Then

\[ \lambda_k \approx \sum_j |P_{jk}|^2 \tau_j^{-2}, \]

and hence

\[ det G = \lambda_1 \ldots \lambda_n \approx \prod_{k=1}^n \left( \sum_j |P_{jk}|^2 \tau_j^{-2} \right). \]

Note that \( \tau_1 = \delta \), and \( \tau_2 = \ldots \tau_{n-1} = \delta^{\frac{1}{2}}, \tau_n = \tau(z', \delta) \gtrsim \delta^{\frac{1}{2}} \).

**Lemma 3.1.** \( det G \approx \prod_{k=1}^n \tau_k^{-2} \).

**Proof.** Let us fix \( \zeta \in V \) for a moment. From (7), we know that \( det G \lesssim \prod_{k=1}^n \tau_k^{-2} \). Let’s prove the reverse inequality. Set \( q_1 = \max \{|P_{1j}|^2; j = 1, 2, \ldots, n\} \). Then \( q_1 \geq \frac{1}{n} \). Without loss of generality, we may assume that \( q_1 = |P_{11}|^2 \), and

\[ |P_{n2}| \leq |P_{n3}| \leq \ldots \leq |P_{nn}|^2. \]

Since \( |P_{11}|^2 \geq \frac{1}{n} \), we have \( |P_{n1}|^2 \leq \frac{n-1}{n} \) and hence \( |P_{nn}|^2 \geq \frac{1}{n(n-1)} \). Since \( |P_{n2}|^2 \leq \frac{1}{n-1} \), we have \( \sum_{k=1}^{n-1} |P_{k2}|^2 \geq 1 - \frac{1}{n-1} \). So

\[ |P_{k2}|^2 \geq \frac{1}{(n-1)^2} \]

for some \( k_2, 1 \leq k_2 \leq n - 1 \). We may assume that \( k_2 = 2 \). Therefore we have

\[ \prod_{k=1}^n \left( \sum_j |P_{jk}|^2 \tau_j^{-2} \right) \geq \left( \frac{1}{n} \tau_2^{-2} \right) \left( \frac{1}{(n-1)^2} \tau_2^{-2} \right) \prod_{k=3}^n \sum_j |P_{jk}|^2 \tau_j^{-2}. \]

Continuing, we have

\[ |P_{nn-1}|^2 \leq \frac{1}{2} \]

and hence

\[ |P_{kn-1n-1}|^2 \geq \frac{1}{2(n-1)} \]
for some \( k_{n-1} \), \( 1 \leq k_{n-1} \leq n - 1 \). So

\[
det G \geq c_n \tau_1^{-2} \tau_2^{-2} \ldots \tau_{n-1}^{-2} \left( \sum_j |P_{jn}|^2 \tau_j^{-2} \right) \geq c_n \prod_{k=1}^n \tau_k^{-2},
\]

because \( |P_{nn}|^2 \geq \frac{1}{n(n-1)} \). Here \( c_n \) is independent of \( z' \), \( \delta \) and \( \zeta \).

□

Set \( G^{-1} = (g^{pq}) \). Then Lemma 3.1 gives us that

\[
(8) \quad |g^{pq}| \lesssim \tau_p \tau_q.
\]

Then the components of the Riemann curvature tensor, \( R^\Omega_{\iota jkl} \), for the Bergman metric are locally defined by

\[
R^\Omega_{\iota jkl} = -\frac{\partial^4}{\partial \zeta_i \partial \bar{\zeta}_j \partial \zeta_k \partial \bar{\zeta}_l} \log K + \sum_{p,q=1}^n g^{pq} \frac{\partial^3}{\partial \zeta_i \partial \zeta_k \partial \bar{\zeta}_p} \log K \frac{\partial^3}{\partial \bar{\zeta}_j \partial \bar{\zeta}_l \partial \zeta_q} \log K
\]

\[
= g_{ij} g_{kl} + g_{il} g_{jk} - \frac{1}{K^2} (KK_{ijkl} - K_{ik} K_{jl})
\]

\[
+ \frac{1}{K^4} \sum_{p,q=1}^n g^{pq} (KK_{ikp} - K_{ik} K_{pq})(KK_{jlp} - K_{jl} K_{pq}).
\]

(9)

Set \( \Delta = (\tau_1, \ldots, \tau_n) \), and \( \Delta^\alpha = \tau_1^{\alpha_1} \ldots \tau_n^{\alpha_n} \), for \( \alpha = (\alpha_1, \ldots, \alpha_n) \). We now estimate all the terms on the right hand side of (9). By (3) and (5), we have

\[
(10) \quad \left| \frac{1}{K^2} (KK_{ijkl}) \right| \lesssim \delta^o \tau^2 (\delta^{n-2} \Delta^{-\alpha}) = \Delta^{-\alpha},
\]

where \( \alpha_i \) denotes the number of \( i \) and \( \bar{i} \), \( 1 \leq i \leq n \), appearing in the subscripts of \( K \). So \( \Delta^{-\alpha} = \tau_1^{\bar{1}} \tau_j^{\bar{1}} \tau_k^{\bar{1}} \tau_l^{\bar{1}} \). Similarly, we have

\[
\left| \frac{1}{K^2} K_{ik} K_{j\bar{l}} \right| \lesssim \Delta^{-\alpha}.
\]

For the terms in the summation, let \( T \) denote any of the terms in the sum. Combining (3), (5), (8) and by the method similar as above, we get

\[
|T(\zeta)| \leq C' \Delta^{-\alpha}.
\]

For the estimates of \( g_{ij} g_{kl} \)'s, we use the estimate (7), and hence we get

\[
(11) \quad |R^\Omega_{\iota jkl}(\zeta)| \leq C \Delta^{-\alpha}.
\]

Let \( Y = \sum_{k=1}^n b_i \frac{\partial}{\partial \zeta_i} \) be a holomorphic tangent vector with unit length,

\[
(12) \quad \sum g_{ij} b_i \bar{b}_j = 1.
\]
Then the holomorphic sectional curvature determined by $Y$ is defined by

$$S(Y) = \sum R^{\Omega}_{ijkl} b_i b_j b_k b_l.$$

Note that (12) and the growth conditions (4), (7) give

$$|b_j(\zeta)| \approx \tau_j,$$

unless $|b_i(\zeta)| = 0$. Hence $|b_i b_j b_k b_l| \leq \Delta^a$. In either case, using (11) and the estimates (13) give $|S(Y)(\zeta)| \leq C$, where the constant $C$ is again independent of $z'$ and $\delta$. To transfer this information to $\Omega$, recall that biholomorphisms are isometries in the Bergman metric, so, as tensors

$$R^{\Omega} = (\Phi^{-1}_{z'})^* R^{\Omega}_{z'}.$$

Thus, if $X = (\Phi_{z'})_* Y$ and $z = \Phi_{z'}(\zeta)$, then

$$|S(X)(z)| \leq C.$$

Finally we vary $z'$ over a small boundary neighborhood of $z_0$ and allow $\delta$ to range in $(0, \delta_0)$. The set of points $z = \Phi_{z'}(\zeta)$ then form a neighborhood of the form $U \cap \Omega$ where $U$ is a neighborhood of $z_0$, and so the proof of the main theorem is complete.

**Remark.** If we extend the domain of the Bergman metric to the set of all complex tangent vectors using the natural almost-complex structure map, the sectional curvature can be explicitly expressed in terms of the holomorphic sectional curvature, and since we know that this is bounded (above and below) it follows that so is the former.

**References**


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