A SIMPLE NASH-MOSER IMPLICIT FUNCTION
THEOREM IN WEIGHTED BANACH SPACES

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Abstract. We prove a simplified version of the Nash-Moser implicit function theorem in weighted Banach spaces. We relax the conditions so that the linearized equation has an approximate inverse in different weighted Banach spaces in each recurrence step.

During the last several decades, “Nash-Moser implicit function theorem” helped to resolve several difficult problems of solvability for nonlinear problems, especially, nonlinear partial differential equations [6,11].

Usually nonlinear partial differential equations (or nonlinear problems in general) can be transformed into solving the problem:

$$\phi(u) = 0$$

where \(\phi\) involves the variables \(x\), the unknown function \(u(x)\) and its derivatives up to the order \(m\).

To prove implicit function theorem in infinite dimensional spaces (as spaces of functions usually are), we first linearize the equation, and then solve the linear equation so that we get the recursive solutions with appropriate recurrence estimates. The simplest one is known as Picard’s iterative scheme. However when \(\phi\) involves the derivatives of \(u\) up to order \(m\), the Picard’s scheme can not be convergent unless the linearized equation gets \(m\) derivatives (as does elliptic equations). To overcome this difficulty, Nash[11] and Moser [10] proposed another scheme involving smoothing operators so that the solution of the linearized equation could be estimated inductively in each Sobolev space of order \(s\). Later Hörmander proposed improved schemes [6,7,8] to get optimal results with respect to the regularity of the solution. However, these schemes are too complicated and rather frightening for the uninitiated reader.

In [12], Saint Raymond established a simplified version of \(C^\infty\) existence theorem so that the number of derivatives that are used (provided that it is finite) does not matter, and this method is useful when we are working on \(C^\infty\) category [1,2,3]. Raymond used the scheme proposed by Moser [10] which consists alternately in using Newton’s scheme and Nash’s smoothing operators closer and closer to the identity:

$$v_k = -\psi(u_k)\phi(u_k), \quad u_{k+1} = u_k + S_kv_k,$$

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and proved the convergence in $C^\infty$ category. Here $S_k$ is a smoothing operator and $\psi(u)$ is the right inverse of $(\partial \phi/\partial u)(u)$, that is,

(1) \[\phi'(u)\psi(u) = I,\]

as an operator ($I =$ identity). Also we need an estimate for $v = -\psi(u)\phi(u)$, so called “tame estimates”;

(2) \[|v|_s \leq C_s(\|\phi(u)\|_{s+d} + |u|_{s+d}\|\phi(u)\|_d),\]

in Hamilton [4].

In some cases, however, we have to deal with the case that a linearized equation has right inverse with error terms of second order, c.f.,[1,2,3], or, in each recurrence step, we have to solve a linearized equation with different weights in each weighted Sobolev space. For example, when we try to prove an embedding problem of a Cauchy-Riemann structure, the (approximate) linearized equation becomes an inhomogeneous Cauchy-Riemann equation on compact pseudoconvex almost complex manifolds (close to being integrable). In this case, we can not get an elliptic regularity for the solution up to the boundary. Therefore, we have to use weighted estimates for $\overline{\partial}$ [5,9] with different weights in each Sobolev space. We then use these weighted estimates for $\overline{\partial}$ (as in (5) below) in each recurrence step in the process of Nash-Moser iteration.

In this paper we prove the Nash-Moser implicit function theorem in weighted Banach spaces. We relax the conditions in (1) and (2) as mentioned above. That is, in each $k$-th recurrence step, $\phi(u) + \phi'(u)(v) = 0$ has a solution with an error that depends on $(\|\phi(u)\|_{3d}^{(k)})^{1+\varepsilon}$, for $\varepsilon > 0$, where $d$ is a positive integer, and can be estimated as

(3) \[|v|_s^{(k)} \leq C_s t_k^{E(s)}(\|\phi(u)\|_{s+d}^{(k)} + |u|_{s+d}\|\phi(u)\|_{d}^{(k)}),\]

where $\{t_k\}$ is an increasing sequence of positive integers and the norms $| \cdot |_{s,\varepsilon}^{(k)}$ (or $\| \cdot \|_{s,\varepsilon}^{(k)}$), defined in each recurrence spaces, are Sobolev norms of order $s$ with weight $e^{-t_k\lambda}$, where $\lambda$ is a $C^\infty$ function, and $E(s)$ is an integer depending on $s$. We state the main theorem as follows.

**Theorem 5.1.** Let $\{t_k\}_{k \geq 0}$ be a strictly increasing sequence of positive integers and suppose that $B^\varepsilon_{s,k}$ and $B^\varepsilon_{s,k}$, $0 \leq s,k \in \mathbb{Z}$, $0 < \varepsilon \in \mathbb{R}$, are families of Banach spaces with the following properties.

(i) For each fixed $k$ and $\varepsilon$, $B^\varepsilon_{s,k} \subset B^\varepsilon_{t,k}$ and $B^\varepsilon_{s,k} \subset B^\varepsilon_{t,k}$ if $s > t$.

(ii) For each fixed $\varepsilon > 0$, and for each fixed $s$, $B^\varepsilon_{s,k_1} \subset B^\varepsilon_{s,k_2}$ and $B^\varepsilon_{s,k_1} \subset B^\varepsilon_{s,k_2}$ if $k_2 \geq k_1$.

(iii) If $| \cdot |_{s,\varepsilon}^{(k)}$ and $\| \cdot \|_{s,\varepsilon}^{(k)}$ denote the norms on $B^\varepsilon_{s,k}$ and $B^\varepsilon_{s,k}$, respectively, then

$| \cdot |_{s,\varepsilon}^{(k)} \geq | \cdot |_{t,\varepsilon}^{(k)}$ and $\| \cdot \|_{s,\varepsilon}^{(k)} \geq \| \cdot \|_{t,\varepsilon}^{(k)}$ if $s > t$.

(iv) For each fixed $\varepsilon > 0$ and $k$, set $B^\varepsilon_{\infty} = \bigcap_{s > 0} B^\varepsilon_{s,k}$ and $B^\varepsilon_{\infty} = \bigcap_{s > 0} B^\varepsilon_{s,k}$. Then there exists an open set $U_0^\varepsilon \subset B^\varepsilon_{\infty}$ with

$U_0^\varepsilon = \{ u \in B^\varepsilon_{\infty}; |u - u_0^{(0)}|_{3d,\varepsilon} < \delta \}$,
where \(d\) is a positive integer, \(u_0^\epsilon \in B_{s0}^{k,0}\), and \(\delta\) is a given positive number. Furthermore, there is a \(C^2\)-map \(\phi_\epsilon : U_0^\epsilon \to B_\infty^{k,0}\) such that if \(u \in U_0^\epsilon\) and \(v, w \in B_\infty^{k,0}\), then

\[
\|\phi_\epsilon(u)\|_{s,d}^{(0)} \leq C_s \epsilon^{-E(s,d)}(1 + |u|_{s+d,d}^{(0)}), \quad s \geq d,
\]

\[
\|\phi_\epsilon'(u)(v)\|_{2d,d}^{(0)} \leq C_1 \epsilon^{-E(d)} |v|_{3d,d}^{(0)},
\]

\[
\|\phi_\epsilon''(u)(v,w)\|_{2d,d}^{(0)} \leq C_2 \epsilon^{-E(d)} |v|_{3d,d}^{(0)}|w|_{3d,d}^{(0)}.
\]

Here \(E(s,d)\) and \(E(d)\) are polynomials in \(s\) and \(d\).

(v) There is a positive number \(a > 0\) with the following properties: for each fixed \(\epsilon > 0\) and for each \(k\), and for all \(u \in U_0^\epsilon\), there exists a linear operator \(\psi_\epsilon^s(u) : B_{s0}^{\infty, k} \to B_{s0}^{\infty, k} + T(a,d) + 2d\), such that

\[
\|\phi_\epsilon(u) - \phi_\epsilon'(u)\psi_\epsilon^s(u)\phi_\epsilon(u)\|_{2d,d}^{(k)} \leq C_1 \left(\|\phi_\epsilon(u)\|_{3d,d}^{(k)}\right)^{1+a},
\]

and \(v_\epsilon^k := -\psi_\epsilon^s(u)\phi_\epsilon(u)\) satisfies, for each \(s, d \leq s \leq k + T(a,d) + 2d\), the following estimates (so called “tame estimates”):

\[
|v_\epsilon^s|_{s,d}^{(k)} \leq C_s \epsilon^{-E_1(s)} t_k^{E_2(s)} \left(\|\phi_\epsilon(u)\|_{s+d,d}^{(k)} + |u|_{s+d,d}^{(0)}\|\phi_\epsilon(u)\|_{s,d}^{(k)}\right),
\]

where \(E_1(s), E_2(s)\) are polynomials in \(s\), \(C_s\) is a constant, and \(T(a,d)\) is an integer, for example, the smallest integer bigger than or equal to \(3d + 3 + \frac{120(d+1)}{a}(2d + 3)\).

(vi) For each \(\epsilon > 0\) and \(k\), there are smoothing operators \(S_\theta : \bigcup_{s=0}^\infty B_s^{\epsilon,k} \to B_{s0}^{\infty,k}\) and \(\tilde{S}_\theta : \bigcup_{s=0}^\infty B_s^{\epsilon,k} \to B_{s0}^{\epsilon,k}\) for all \(\theta > 1\) such that for each real number \(s, t\), there are a constant \(C_{s,t}\) and integers \(E_1(s,t)\) and \(E_2(s,t)\) such that

\[
|S_\theta v|_{s,t}^{(k)} \leq C_{s,t} \epsilon^{-E_1(s,t)} t_k^{E_2(s,t)} \theta^{s-t} |v|_{s,t}^{(k)}, \quad s \geq t,
\]

\[
|v - S_\theta v|_{s,t}^{(k)} \leq C_{s,t} \epsilon^{-E_1(s,t)} t_k^{E_2(s,t)} \theta^{s-t} |v|_{s,t}^{(k)}, \quad s \leq t,
\]

and the similar estimates hold for \(\tilde{S}_\theta\).

Then there exist an integer \(D\) and a small number \(b > 0\) such that if \(\|\phi_\epsilon(u_0)\|_{D}^{(0)} < b\), for some \(\epsilon > 0\) and for some \(u_0^\epsilon \in U_0^\epsilon\), then there exists an element \(u \in U_0^\epsilon\) such that \(\phi_\epsilon(u) = 0\).

Remark 5.2. (a) In many cases, we approximate the non-linear problem up to second order errors. Hence \(a = 1\) in these cases.

(b) If \(\lambda\) is a smooth bounded function, we can modify \(\lambda\) so that \(1 \leq \lambda \leq 1 + \frac{a}{4}\). Let \(B_s^k, B_s^k\) be the weighted Sobolev spaces of order \(s\) on a bounded domain \(\Omega \subset \mathbb{C}^n\) with weighted norm

\[
(\|f\|_{s}^{(k)})^2 = \sum_{|\alpha| \leq s} \int_{\Omega} |D^\alpha f|^2 e^{-t_k \lambda} dV, \quad f \in B_s^k,
\]

then (5.3) holds with \(\theta_0 = e\).
Theorem 1. Let \( \{t_k\}_{k \geq 0} \) be a strictly increasing sequence of positive integers and suppose that \( B^k_s \) and \( B^k_k \), \( s, k \geq 0 \), are families of Banach spaces with the following properties.

(i) For each fixed \( k \), \( B^k_s \subset B^k_t \) and \( B^k_s \subset B^k_t \) if \( s > t \).

(ii) If \( |\cdot|^{(k)}_s \) and \( \|\cdot\|^{(k)}_s \) denote the norms on \( B^k_s \) and \( B^k_s \), respectively, then

\[
|\cdot|^{(k)}_t \geq |\cdot|^{(k)}_s \quad \text{and} \quad \|\cdot\|^{(k)}_t \geq \|\cdot\|^{(k)}_s \quad \text{if} \quad s > t.
\]

(iii) For each fixed \( k \), if \( B^k_\infty = \bigcap_{s > 0} B^k_s \) and \( B^k_\infty = \bigcap_{s > 0} B^k_s \), then there exists an open set \( U_k \subset B^k_\infty \) with

\[
U_k = \{ u \in B^k_\infty : |u - u_0|^{(k)}_{3d} < \delta \},
\]

where \( d \) is a positive integer, \( u_0 \in B^0_\infty \), and \( \delta \) is a given positive number. Furthermore, there is a \( C^2 \)-map \( \phi : U_k \rightarrow B^k_\infty \) such that if \( u \in U_k \) and \( v, w \in B^k_\infty \), then

\[
\left| \phi'(u)(v) \right|^{(k)}_2 \leq C_1 |v|^{(k)}_{3d},
\]

\[
\left| \phi''(u)(v, w) \right|^{(k)}_2 \leq C_2 |v|^{(k)}_{3d} |w|^{(k)}_{3d},
\]

(when one deals with (nonlinear) partial differential equations of order \( m \), these estimates classically hold for \( d > m + n/2 \)).

(iv) There is a positive number \( \epsilon > 0 \) with the following properties: for each \( k \), and for all \( u \in U_k \), there exists a linear operator \( \psi_k(u) : B^k_\infty \rightarrow B^k_{k+T(\epsilon, d)+2d} \) such that

\[
\left| \psi_k(u) \phi(u) - \psi_k(u) \phi(u) \right|^{(k)}_{2d} \leq C_1 \left( \|\phi(u)\|^{(k)}_{3d} \right)^{1+\epsilon},
\]

and \( v_k := -\psi_k(u) \phi(u) \) satisfies, for each \( s, d \leq s \leq k + T(\epsilon, d) + 2d \), the following estimates (so called “tame estimate”):

\[
|v_k|^{(k)}_s \leq C_s t_k^{E(s)} \left( \|\phi(u)\|^{(k)}_{s+d} + |u|^{(k)}_{s+d} \|\phi(u)\|^{(k)}_{d} \right),
\]

where \( E(s) \) is a polynomial in \( s \), \( C_s \) is a constant, and \( T(\epsilon, d) \) is an integer, for example, the smallest integer bigger than or equal to \( 3d + 3 + \frac{120(d+1)}{\epsilon}(2d + 3) \).

(v) There is \( \theta_0 > 1 \) such that

\[
\theta_0^{(1+\epsilon)(t_k-t_{k+1})} |\cdot|^{(k)}_s \leq |\cdot|^{(k+1)}_s \leq \theta_0^{t_k-t_{k+1}} |\cdot|^{(k)}_s,
\]

\[
\theta_0^{(1+\epsilon)(t_k-t_{k+1})} \|\cdot\|^{(k)}_s \leq \|\cdot\|^{(k+1)}_s \leq \theta_0^{t_k-t_{k+1}} \|\cdot\|^{(k)}_s,
\]

where \( \epsilon > 0 \) is the number in (iv) satisfying (4), and hence \( B^k_s \subset B^k_{s+1} \), \( B^k_s \subset B^k_{s+1} \), for each \( s \geq 0 \).
For each $k$, there are smoothing operators $S_\theta : \bigcup_{s=0}^{\infty} B_s^k \to B_s^k$ and $\tilde{S}_\theta : \bigcup_{s=0}^{\infty} B_s^k \to B_s^k$ for all $\theta > 1$ such that for each real number $s, t$, there are a constant $C_{s,t}$ and an integer $E(s,t)$ such that

\[
|S_\theta v|^{(k)}_s \leq C_{s,t} \theta^{s-t} |v|^{(k)}_t, \quad s \geq t,
\]

\[
|v - S_\theta v|^{(k)}_s \leq C_{s,t} \theta^{s-t} |v|^{(k)}_t, \quad s \leq t,
\]

and the similar estimates hold for $\tilde{S}_\theta$.

Then there exist an integer $B$ and a small number $b > 0$ such that if $\|\phi(u_0)\|^0_B < b$, for some $u_0 \in U_0$, then there exists an element $u \in U_0$ such that $\phi(u) = 0$.

Remark 2. (a). Since $\|\phi(u)\|^{(k)}_{3d} \leq 1$, we may assume that $0 < \epsilon \leq 1$.

(b). In many cases, we approximate the non-linear problem up to second order error terms, and hence $\epsilon = 1$ in these cases.

(c). If $\lambda$ is a smooth bounded function, we can modify $\lambda$ so that $1 \leq \lambda \leq 1 + \frac{\epsilon}{4}$. Let $B_s^k$ be the weighted Sobolev space of order $s$ on a bounded domain $\Omega \subset \mathbb{C}^n$ with weighted norm:

\[
(\|f\|^{(k)}_s)^2 = \sum_{|\alpha| \leq s} \int_\Omega |D^\alpha f|^2 e^{-t_k \lambda} dV, \quad f \in B_s^k.
\]

Then (6) holds with $\theta_0 = e$.

By choosing a subsequence if necessary, we may assume that the sequence $\{t_k\}_{k \geq 0}$ satisfies

\[
t_{k+1} \geq \frac{3}{2} t_k, \quad k \geq 0,
\]

and $t_0$ is sufficiently large. Also, we will use a sequence of real numbers $\{\theta_k\}$ defined inductively as follows:

\[
\theta_k = \theta_0^{t_k}, \quad k \geq 1,
\]

and will use the corresponding smoothing operators $S_{\theta_k}$. In the sequel, we set

\[
\tau_0 = (1 + 2\epsilon/3) > 1, \quad \text{and} \quad \tau_k = 1 + \frac{(10 - k)}{15} \epsilon, \quad k = 1, 2, \ldots, 5,
\]

and hence $1 < \tau_5 < \tau_4 < \tau_3 < \tau_2 < \tau_1 < \tau_0$. For a convenience, we set

\[
T := T(\epsilon, d) := [3d + 3 + \frac{120(d + 1)}{\epsilon} (2d + 3)],
\]

where $[\Gamma]$ denotes the smallest integer bigger than or equal to $\Gamma$. We first prove the following Lemma which is a crucial step in the proof of Theorem 1.
Lemma 3. With the same assumption as in the theorem and with the smoothing operators \( S_{\theta_k} \) of the remark, the sequences

\[ v_k = -\psi_k(u_k)\phi(u_k), \quad u_{k+1} = u_k + S_{\theta_{k+1}}v_k, \]

are well defined if \( \|\phi(u_0)\|^{(0)}_{2d} \leq \theta_{0}^{-2t_0} \) for sufficiently large \( t_0 \); more precisely, there exist constants \( (U_i)_{i \geq d} \), and \( V \) (independent of \( k \)) such that for \( k \geq 0 \),

\[
\begin{align*}
(i)_k & \quad |u_k - u_0|^{(k)}_{3d} < \delta \text{ and } \|\phi(u_k)\|^{(k)}_{2d} \leq \theta_k^{-\tau_0}, \\
(ii)_k & \quad |v_k|^{(k)}_{3d+3} \leq V \theta_k^{-\tau_3}, \\
(iii)_k & \quad (1 + |u_{k+1}|^{(k+1)}_{s+2d}) \leq U_s \theta_{k+1}^{2d+1/2} (1 + |u_k|^{(k)}_{s+2d}), \quad d \leq s \leq k + T + 2d.
\end{align*}
\]

Proof. Since the property \((i)_k\) implies that the sequence \( u_k \) and \( v_k \) are well defined, it is sufficient to prove \((i)_k\), \((ii)_k\) and \((iii)_k\) inductively. The property \((i)_0\) is true by assumption.

Proof of \((ii)_{k+1}\). The same estimate (5) gives, for every \( s, d \leq s \leq k + T + 2d \), that

\[ |v_k|^{(k)}_{s} \leq C_s t_k^{E(s)} \left( \|\phi(u_k)\|^{(k)}_{s+d} + |u_k|^{(k)}_{s+d} \|\phi(u_k)\|^{(k)}_{2d} \right). \]

For \( s = d \), and using \((i)_k\) and (10), we have

\[ |v_k|^{(k)}_{d} \leq C_d t_k^{E(d)} \left( 1 + |u_k - u_0|^{(k)}_{2d} + |u_0|^{(k)}_{2d} \right) \|\phi(u_k)\|^{(k)}_{2d} \leq V_0 \theta_k^{-\tau_1}, \]

because \( t_k^{E(d)} \theta_k^{1-\tau_0} \) is bounded.

Let \( T \) be the number defined in (9) and set \( N = 4(2d+1) \). From the tame estimate (5) and the properties of \( \|\phi(u_k)\|^{(k)}_{s} \) stated in \((iii)\) of Theorem 1, it follows, for \( d \leq s \leq k + T + 2d \), that

\[
\begin{align*}
|v_k|^{(k)}_{s} & \leq C_s t_k^{E(s)} \left( \|\phi(u_k)\|^{(k)}_{s+d} + |u_k|^{(k)}_{s+d} \|\phi(u_k)\|^{(k)}_{d} \right) \\
& \leq C_s t_k^{E(s)} \left( C_{s+d} (1 + |u_k|^{(k)}_{s+2d}) + C_d (1 + |u_k|^{(k)}_{2d}) |u_k|^{(k)}_{s+2d} \right) \\
& \leq C_s t_k^{E(s)} \left( C_{s+d} + C_{2d} (1 + \delta + |u_0|^{(0)}_{3d}) \right) \cdot (1 + |u_k|^{(k)}_{s+2d}).
\end{align*}
\]

The estimate

\[ (1 + |u_j|^{(j)}_{T+2d}) \leq (1 + |u_0|^{(0)}_{T+2d}) \theta_j^{N-\frac{1}{4}} \]

holds obviously for \( j = 0 \). Moreover, if it holds for some \( j < k \), we obtain from (7), (13) and \((iii)_j\) that

\[
\begin{align*}
(1 + |u_j|^{(j+1)}_{T+2d}) & \leq U_T \theta_{j+1}^{2d+1/2} (1 + |u_j|^{(j)}_{T+2d}) \\
& \leq U_T \theta_{j+1}^{-1/4} (1 + |u_0|^{(0)}_{T+2d}) \theta_j^{N} \theta_{j+1}^{2d+1-\frac{1}{4}} \\
& \leq U_T \theta_{j+1}^{-1/4} (1 + |u_0|^{(0)}_{T+2d}) \theta_j^{N} \theta_{j+1}^{-\frac{1}{4}}.
\end{align*}
\]
because \( \theta_j^N \leq \theta_{j+1}^{2N/3} \) and \((2d+1) + 2N/3 < N\). Therefore, by induction for \(j \leq k\) that, (13) holds provided \(t_0\) (and hence \(t_1\)) is sufficiently large so that \(\theta_0^{-1/4} U_T \leq 1\).

To thanks to (13), we may write (12) as:

\[
|v_k|^{(k)}_{3d+3} \leq |S_{\theta_k} v_k^{(k)}_{3d+3} + v_k - S_{\theta_k} v_k^{(k)}_{3d+3} \\
\leq C_{3d+3, d} t_k^{E(3d+3, d)} \theta_{k}^{2d+3} v_k^{(k)}_{3d+3} + C_{3d+3, T} t_k^{E(3d+3, T)} \theta_{k}^{3d+3-T} v_k^{(k)}_{T} \\
\leq C_{3d+3, d} V_0 t_k^{E(3d+3, d)} E(d) \theta_{k}^{-\tau_2} + C_{3d+3, T} V_1 t_k^{E(3d+3, T)} \theta_{k}^{-4(2d+2)+N} \\
\leq V \theta_{k}^{-\tau_3}.
\]

Because \(t_k^{E(3d+3, T)} \theta_{k}^{-4(2d+2)+N+\tau_3}\) and \(t_k^{E(3d+3, d)} E(d) \theta_{k}^{-\tau_2+\tau_3}\) are bounded. This proves (ii)_{k+1}.

Proof of (iii) Now we want to estimate \(|u_{k+1}|_{s+2d}^{(k+1)}\) in terms of \(|v_k|_{s}^{(k)}\). Since \(u_{k+1} = u_k + S_{\theta_{k+1}} v_k\), it follows, for \(d \leq s \leq k + T\), that

\[
|u_{k+1}|_{s+2d}^{(k+1)} \leq |u_k|_{s+2d}^{(k)} + |S_{\theta_{k+1}} v_k|_{s+2d}^{(k+1)} \leq |u_k|_{s+2d}^{(k)} + C_{s+2d, d} t_k^{E(s+2d, s)} \theta_{k+1}^{2d} |v_k|_{s}^{(k)}.
\]

Thus one obtains from (13) that

\[
1 + |u_{k+1}|_{s+2d}^{(k+1)} \leq W_s t_k^{E(s+2d, s) + E(s)} \theta_{k+1}^{2d} (1 + |u_k|_{s+2d}^{(k)}) \\
\leq U_s \theta_{k+1}^{2d+1/2} (1 + |u_k|_{s+2d}^{(k)}),
\]

because \(t_k^{E(s+2d, s) + E(s)} \theta_{k+1}^{-\frac{1}{2}}, d \leq s \leq k + T + 2d\), is bounded. This proves (iii)_{k} with constants \(U_s\) does not depend on \(k\).

Proof of (i)_{k+1}. Since \(u_k - u_0 = \sum_{j<k} S_{\theta_{j+1}} v_j, \forall t \in [0, 1]\), one can write, from (6) and (ii)_{k}, that

\[
|u_k + t S_{\theta_{k+1}} v_k - u_0|_{3d}^{(k+1)} \leq \sum_{j \leq k} |S_{\theta_{j+1}} v_j|_{3d}^{(k+1)} \\
\leq C_{3d, 3d} \theta_{0}^{t_k - t_{k+1}} t_k^{E(3d, 3d)} \sum_{j \leq k} |v_j|_{3d}^{(j)} \\
\leq C_{3d, 3d} V \theta_{0}^{-t_{k+1}/3} t_k^{E(3d, 3d)} \sum_{j \leq k} \theta_{j}^{-\tau_3} \\
\leq C_{3d, 3d} V S \theta_{0}^{-t_{k+1}/4},
\]

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where \( S = \sum_{j \leq \infty} \theta_j^{-\tau_3} < \infty \) is a constant. By choosing \( t_0 \) (and hence \( t_1 \)) sufficiently large so that \( C_{3d, 3d} V S \theta_0^{-t_1/4} < \delta \), we have
\[
|u_{k+1} - u_0|_{3d}^{(k+1)} = |\sum_{j < k+1} S_{\theta_j} v_j^{(k+1)}|_{3d} < \delta,
\]
and this is the first part of \((i)_{k+1}\).

By virtue of Taylor’s formula, we can write:
\[
\phi(u_{k+1}) = \phi(u_k) + \phi'(u_k)S_{\theta_{k+1}} v_k
+ \int_0^1 (1 - t)\phi''(u_k + tS_{\theta_{k+1}} v_k)(S_{\theta_{k+1}} v_k, S_{\theta_{k+1}} v_k)dt
= \phi_1 + \phi_2 + \phi_3,
\]
where
\[
\phi_1 = \phi(u_k) + \phi'(u_k) v_k
\phi_2 = \phi'(u_k)(S_{\theta_{k+1}} v_k - v_k)
\phi_3 = \int_0^1 (1 - t)\phi''(u_k + tS_{\theta_{k+1}} v_k)(S_{\theta_{k+1}} v_k, S_{\theta_{k+1}} v_k)dt.
\]

First, we estimate \( \phi_1 \). For this, we use the following two estimates:
\[
\|\phi(u_k)\|_{2d}^{(k)} \leq \theta_k^{(1+2\varepsilon)/3} = \theta_k^{-\tau_0} \quad \text{and} \quad \|\phi(u_k)\|_{T+d}^{(k)} \leq A\theta_k^N.
\]

Note that the second inequality comes from the properties (iii) in Theorem 1 and (14) with \( s = T + d \). From (4), we have \( \|\phi_1\|_{2d}^{(k+1)} = \|\phi(u_k) + \phi'(u_k)(v_k)\|_{2d}^{(k+1)} \leq C_1(\|\phi(u_k)\|_{3d}^{(k+1)})^{1+\varepsilon} \).

Setting \( \bar{\theta}_k = \theta_k^{\frac{2\varepsilon}{3}} \), it follows from (16) that
\[
\|\phi(u_k)\|_{3d}^{(k)} = \|\tilde{S}_{\theta_k} \phi(u_k)\|_{3d}^{(k)} + \|\phi(u_k) - \tilde{S}_{\theta_k} \phi(u_k)\|_{3d}^{(k)}
\leq C_{3d, 2d} t_k^{E(3d, 2d)} \tilde{\theta}_k^{E(3d, T+d)} \|\phi(u_k)\|_{2d}^{(k)} + C_{3d, T+d} t_k^{E(3d, T+d)} \bar{\theta}_k^{3d - T - d} \|\phi(u_k)\|_{T+d}^{(k)}
= C_{3d, 2d} t_k^{E(3d, 2d)} \bar{\theta}_k^{-1} + C_{3d, T+d} t_k^{E(3d, T+d)} A\bar{\theta}_k^{-1}
= (C_{3d, 2d} t_k^{E(3d, 2d)} + C_{3d, T+d} t_k^{E(3d, T+d)} A)\bar{\theta}_k^{-1}.
\]

Hence it follows from (6) that
\[
\|\phi_1\|_{2d}^{(k+1)} \leq C_1(\|\phi(u_k)\|_{3d}^{(k+1)})^{1+\varepsilon}
= C_1 \theta_0^{(1+\varepsilon)(t_k - t_{k+1})}(\|\phi(u_k)\|_{3d}^{(k+1)})^{1+\varepsilon}
\leq C_1 \left( C_{3d, 2d} t_k^{E(3d, 2d)} + C_{3d, T+d} t_k^{E(3d, T+d)} A \right)^{1+\varepsilon} \theta_0^{(1+\varepsilon)(t_k - t_{k+1})} \theta_k^{-(1+\varepsilon)}
\leq C_1 \left( C_{3d, 2d} t_k^{E(3d, 2d)} + C_{3d, T+d} t_k^{E(3d, T+d)} A \right)^{1+\varepsilon} \theta_0^{-(t_{k+1})(1+\varepsilon)}
\leq \frac{1}{3} \theta_k^{-(1+2\varepsilon/3)},
\]
by choosing $t_0$ sufficiently large.

Next, let us estimate $\phi_2$. From (6), (15) and the property (iii) of Theorem 1, we obtain that

$$
\|\phi_2\|^{(k+1)}_{2d} \leq \theta_0^{t_k-t_{k+1}} C_1 |S_{\theta_{k+1}} - v_k|^{(k)}_{3d} \\
\leq C_1 C_{3d,3d+3} \epsilon(t_k^{(3d,3d+3)}) \theta_{k+1}^{-3} v_k^{(k)}|_{3d} \theta_0^{t_k-t_{k+1}} \\
\leq C_1 C_{3d,3d+3} V_k^{(3d,3d+3)} \theta_{k+1}^{-3} \theta_k^{-\tau_3} \theta_0^{t_k-t_{k+1}} \\
\leq \frac{1}{3} \theta_k^{-(1+2\epsilon/3)},
$$

provided $t_0$ is sufficiently large.

Finally, we estimate $\phi_3$. By choosing $t_0$ sufficiently large, we have, from (15), that

$$
\|\phi_3\|^{(k+1)}_{2d} \leq (|S_{\theta_{k+1}} v_k|^{(k+1)}_{3d})^2 \leq \theta_0^{2(t_k-t_{k+1})} (|S_{\theta_{k+1}} v_k|^{(k)}_{3d})^2 \\
\leq \left(C_{3d,3d+3} \epsilon(t_k^{(3d,3d+3)}) \theta_{k+1}^{-3} v_k^{(k)}|_{3d} \right)^2 \theta_0^{2(t_k-t_{k+1})} \\
\leq \left(C_{3d,3d+3} V_k^{(3d,3d+3)} \right)^2 \theta_{k+1}^{-2} \theta_0^{2(t_k-t_{k+1})} \theta_k^{-2\tau_3} \\
\leq \frac{1}{3} \theta_k^{-(1+2\epsilon/3)}.
$$

If we combine the estimates of $\phi_1$, $\phi_2$ and $\phi_3$, the second part of $(i)_{k+1}$ follows.

We define a constant $M = \frac{\tau_4 + N + 1}{\tau_3 - \tau_4} = \frac{15}{13} \epsilon + 4(2d + 1) + 1$, and for each integer $k \geq 0$, set

$$
\Lambda(k) = \frac{k + T + (2 + M)d - 1}{M + 1}.
$$

The proof of the property $(ii)_k$ of Lemma 3 can be modified to prove an estimate for $|v_k|^{(k)}_{s}$ for every $s \geq d$.

**Lemma 4.** There exist constants $(V_s)_{s \geq d}$ such that the sequence $\{v_k\}$ of Lemma 3 satisfies, for every $k \geq 0$ and $d \leq s \leq \Lambda(k)$, that

$$
|v_k|^{(k)}_{s} \leq V_s t_k^{\max(E(s,d),E(s,D))} \theta_k^{-\tau_4},
$$

where $D = \left\lceil \frac{(s-d)}{\tau_3 - \tau_4} (\tau_4 + N + 1) + s \right\rceil$.

**Proof.** Keeping the value $N = 4(2d + 1)$, we obtain from (7) and $(iii)_k$ of Lemma 3 that

$$
(1 + |u_k|^{(k+1)}_{s+2d}) \theta_k^{-N} \leq U_s \theta_k^{2d+1/2} (1 + |u_k|^{(k)}_{s+2d}) \theta_k^{-N} \\
\leq U_s \theta_k^{-1/2} (1 + |u_k|^{(k)}_{s+2d}) \theta_k^{-N+2d+1} \\
\leq U_s \theta_k^{-1/2} (1 + |u_k|^{(k)}_{s+2d}) \theta_k^{-N},
$$

where $U_s$ is a constant depending on $s$ and $s$. This completes the proof of Lemma 4.
for \( d \leq s \leq k + T + 2d \). Therefore for each fixed \( s \), the sequence \((1 + \lvert u_k \rvert_{s+2d}) \theta_k^{-N}\) is bounded, and hence there exists a constant \( K_s \) such that

\[
(1 + \lvert u_k \rvert_{s+2d}) \theta_k^{-N} \leq K_s.
\]

Substituting this into (13), we obtain, for \( d \leq s \leq k + T + 2d \), that

\[
\| v_k \|_{s}^{(k)} \leq C_s \{ (C_{s+d} + C_{2d} (1 + \delta + \lvert u_0 \rvert_{(0)}))K_s t_k^{E(s)} \theta_k^N = W_s t_k^{E(s)} \theta_k^N \leq W_s \theta_{k+1}^N + 1,
\]

where \( N = 4(2d + 1) \) does not depend on \( s \). Now, if \( d \leq s \leq \Lambda(k) \), the definition of \( \Lambda(k) \) in (17) shows that \( D \leq k + T + 2d \). Therefore for \( d \leq s \leq \Lambda(k) \), we rewrite our interpolation formula with \( \bar{\theta}_k = \theta_k^{(c_3 - c_4)} \);

\[
\| v_k \|_{s}^{(k)} \leq \| S_{\bar{\theta}_k} v_k \|_{s} + \| v_k - S_{\bar{\theta}_k} v_k \|_{s}^{(k)} \\
\leq C_s d_k t_k^{E(s,d)} \bar{\theta}_k^{-d} \| v_k \|_{d}^{(k)} + C_s d_k t_k^{E(s,d)} \bar{\theta}_k^{-d} \| v_k \|_{d}^{(k)} \\
\leq C_s d_k t_k^{E(s,d)} V \theta_k^{-\tau} + C_s d_k t_k^{E(s,d)} W \theta_k^{-\tau} \\
\leq V s_{t_k}^{\max(E(s,d),E(s,d))} \theta_k^{-\tau},
\]

where we have used the estimate (18).

**Proof of the Theorem 1** Let \( u_k \) and \( v_k \) be as in Lemma 3. From Lemma 4 we have, for any \( j \geq 0 \) and \( d \leq s \leq \Lambda(j) \), that

\[
\| S_{\bar{\theta}_{j+1}} v_j \|_{s}^{(j)} \leq C_s d_k t_k^{E(s)} \| v_j \|_{s}^{(j)} \leq C_s d_k t_k^{E(s,s)+\max(E(s,d),E(s,d))} \theta_j^{-\tau} \leq A_s \theta_j^{-\tau_5}.
\]

By (6) one thus obtains, for \( d \leq s \leq \Lambda(j) \), that

\[
\| S_{\bar{\theta}_{j+1}} v_j \|_{s}^{(0)} \leq \theta_0^{1(1+\epsilon/4)(t_j - t_0)} \| S_{\bar{\theta}_{j+1}} v_j \|_{s}^{(0)} \\
\leq C_s d_k t_k^{E(s,s)+\max(E(s,d),E(s,d))} \theta_j^{-\tau} \\
= C_s d_k t_k^{E(s,s)+\max(E(s,d),E(s,d))} \theta_j^{-\tau} \\
\leq C_s d_k t_k^{E(s,s)+\max(E(s,d),E(s,d))} \theta_j^{-\tau}.
\]

because \((1 + \epsilon/4) - \tau_5 = -\epsilon/12\). By virtue of (17), we also have

\[
\Lambda(j) < s \text{ if and only if } j < \Lambda^{-1}(s) = s(M + 1) - (2 + M)d - T + 1.
\]

Now for each fixed \( s \geq 0 \), the sequence \( u_k = u_0 + \sum_{j<k} S_{\theta_{j+1}} v_j \) is convergent in \( B_s^0 \) because

\[
\| S_{\bar{\theta}_{j+1}} v_j \|_{s}^{(0)} \leq \| S_{\bar{\theta}_{j+1}} v_j \|_{s}^{(0)} + \sum_{j \geq \Lambda^{-1}(s)} \| S_{\bar{\theta}_{j+1}} v_j \|_{s}^{(0)} < \infty,
\]

where the first sum in the right is finite sum by (20), and the second sum in the right is finite by (19). Moreover, the limit \( u \in B_s^0 \) of the sequence \( u_k \) satisfies

\[
\| \phi(u) \|_{2d}^{(0)} \leq \| \phi(u_k) \|_{2d}^{(0)} + \| \int_0^1 \phi'(u_k + t(u - u_k))(u - u_k)dt \|_{2d}^{(0)} \\
\leq \| \phi(u_k) \|_{2d}^{(0)} + C_1 |u - u_k|_{3d}^{(0)},
\]

and by (6), one obtains that

\[
\| \phi(u_k) \|_{2d}^{(0)} \leq \theta_0^{1+\epsilon/4}(t_k - t_0) \| \phi(u_k) \|_{2d}^{(k)} \leq \theta_k^{-5\epsilon/12},
\]

for all \( k \). Therefore, by taking limit for \( k = \infty \), it follows that \( \phi(u) = 0 \) and this proves Theorem 1 with \( B = 2d \) and \( b = \theta_0^{-2t_0} \).
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