Local extension of boundary holomorphic forms on real hypersurfaces in $\mathbb{C}^n$

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Abstract

Let $\mathcal{M}$ be a real hypersurface in $\mathbb{C}^n$ with $n \geq 3$ and the Levi-form at $p \in \mathcal{M}$ has $(q+1)$-positive eigenvalues, $1 \leq q \leq n-2$. We solve one-sided local $\partial$-closed extension problem near $p$ for $(0,r)$-forms for $1 \leq r \leq q$.

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1 Introduction

Let $\mathcal{M}$ be a real hypersurface in $\mathbb{C}^n$ with a smooth defining function $\rho$. The Cauchy-Riemann complex on $\mathbb{C}^n$ induce in a natural way the tangential Cauchy-Riemann complex or $\overline{\partial}b$ complex on $\mathcal{M}$:

$$0 \rightarrow B^{p,0}(\mathcal{M}) \xrightarrow{\overline{\partial}b} B^{p,1}(\mathcal{M}) \xrightarrow{\overline{\partial}b} \cdots \xrightarrow{\overline{\partial}b} B^{p,n-1}(\mathcal{M}) \rightarrow 0.$$  

In the above complex $B^{p,q}(\mathcal{M})$ consists of the restriction of smooth $(p,q)$ forms in $\mathbb{C}^n$ to $\mathcal{M}$ which are pointwise orthogonal to the ideal generated by $\partial\rho$. Let $p \in \mathcal{M}$ be a fixed point and $U$ be a neighborhood of $p$ in $\mathbb{C}^n$ where $\rho$ is defined. Set $U^- = \{z \in U : \rho(z) \leq 0\}$ and $U^+ = \{z \in U : \rho(z) \geq 0\}$. We denote the smooth $(p,q)$ forms on $W$ by $\Lambda^{p,q}(W)$. For $\alpha \in B^{p,q}(\mathcal{M} \cap U)$ with $\overline{\partial}b\alpha = 0$, if there exists a smooth $(p,q)$-form $\tilde{\alpha} \in \Lambda^{p,q}(U^-)$ with $\overline{\partial}\tilde{\alpha} = 0$ in $U^-$ and $(\tilde{\alpha} - \alpha) \wedge \partial\rho = 0$ on $\mathcal{M} \cap U$, we call $\tilde{\alpha}$ a local one-sided weak $\overline{\partial}$-closed extension of $\alpha$. Moreover if $\tilde{\alpha}$ satisfies $\overline{\partial}\tilde{\alpha} = 0$ in $U$, we say that $\tilde{\alpha}$ is a two-sided extension of $\alpha$.

When $\mathcal{M}$ is the boundary of a smoothly bounded domain $\Omega$ in $\mathbb{C}^n$, the global $\overline{\partial}$-closed extension problem for forms from $\mathcal{M}$ to the domain $\Omega$ was studied by J.J. Kohn and H. Rossi [7] who first introduced the $\overline{\partial}b$ complex. They showed that a global $\overline{\partial}$-closed extension exists for any $(p,q)$-form from the boundary $\mathcal{M} = b\Omega$ to the domain $\Omega$ in a complex manifold if $\Omega$ satisfies the condition $Z(n - q - 1)$ at all points of $b\Omega$. Analogous result was obtained by Henkin and Leiterer [6] using kernel methods. When $\Omega$ is a bounded pseudoconvex domain in $\mathbb{C}^n$, Shaw and Boas [10, 12] constructed a two-sided $\overline{\partial}$-closed extension for $\overline{\partial}b$-closed forms near $b\Omega$, using the $L^2$-Cauchy problem for $\overline{\partial}$ constructed by Shaw [11], and solved $\overline{\partial}b$ problem on the boundary with Sobolev estimates.

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Laurent-Thiébaut and Leiterer [9] showed that if a bounded domain $\Omega$ is defined by a global $(q + 1)$-convex defining function and there is a closed set $K \subseteq \Omega$ defined by a $(q + 2)$-convex function, then for a Hörder continuous $\partial_b$-closed form on $V = b\Omega \setminus K$, there exists a Hörder continuous $\partial$-closed extension into $\Omega \setminus K$. However these assumptions are global, in a sense, because $\mathcal{M}$ is the boundary of a bounded domain which is defined by a global $(q + 1)$-convex defining function. Furthermore, the necessary condition of the existence of the closed set $K$ is global and seems not intrinsic.

For the local extension problem, Anreotti and Hill [1] solved the local one-sided weak $\bar{\partial}$-closed extension problem when the Levi-form at the reference point has $(0, q)$ forms, $q \geq 1$, exists if $H^{p,q+1}(U^{-}, T) = 0$ in part I of [1], and then showed that $H^{p,q+1}(U^{-}, T) = 0$ when the Levi-form at $p \in M$ has $(q + 2)$ positive eigenvalues in part II of [1].

However, it seems that a sufficient condition for the existence of local extension is $(q + 1)$ positive eigenvalues. In this paper, we develop a method to prove the local extension problem which uses the smooth solvability of $\bar{\partial}$ equation on parameters. Then we prove the local $\bar{\partial}$-closed extension problem when the Levi form at the reference point has $(q + 1)$ positive eigenvalues:

**Theorem 1.1.** Let $\mathcal{M}$ be a real hypersurface in $C^n$, $n \geq 3$, and suppose that the Levi-form at $p \in M$ has $(q + 1)$ positive eigenvalues, $1 \leq q \leq n - 2$. Then there is a neighborhood $U$ of $p$ such that for any $\alpha \in \wedge^{0,r}(\mathcal{M} \cap U)$, $1 \leq r \leq q$, satisfying $\partial_\alpha = 0$ on $\mathcal{M} \cap U$, there exists $\tilde{\alpha} \in \wedge^{0,r}(U^{-})$ such that $\partial \tilde{\alpha} = 0$ in $U^{-} \setminus \mathcal{M}$ and $(\tilde{\alpha} - \alpha) \wedge \partial \tilde{\rho} = 0$ on $\mathcal{M} \cap U$.

**Remark.** (1). The condition in the Main Theorem is local and satisfies the condition $Z(n-q-1)$ at $p \in M$, which is an intrinsic property at $p \in \mathcal{M}$.

(2). In [8], Xu proved the local one-sided $\bar{\partial}$-closed extension problem for $(0,1)$ forms except some bad set $B \subset \mathcal{M}$.

Note that the local $\bar{\partial}$-closed extension problem and the local solvability of $\partial_b$ equation are closely related [10, 12]. Following the method which is developed in the proof of the Main theorem, we can solve a local $\partial_b$ problem in certain cases.

**Corollary 1.2.** Let $\mathcal{M}$ be a real hypersurface whose Levi form at $p \in M$ has max$(q + 1, n - q)$ positive eigenvalues, $1 \leq q \leq n - 2$. Then there exists a neighborhood $U$ of $p$ such that for $\alpha \in \mathcal{B}^{0,q}(\mathcal{M} \cap U)$ with $\partial_\alpha = 0$, there exists $u \in \mathcal{B}^{0,q-1}(\mathcal{M} \cap U)$ such that $\partial_b u = \alpha$.

**Remark.** We say the condition $Y(q)$ is satisfied at $p$ if the Levi form at $p$ has either max$(q + 1, n - q)$ positive eigenvalues or min$(q + 1, n - q)$ positive and negative eigenvalues. Andreotti and Hill [1] and Shaw [13] solved the local $\partial_b$ problem when the condition $Y(q)$ is satisfied. Therefore our result in Corollary 1.2 is already contained in [1, 13] while we suggest a different method to prove the local solvability of $\partial_b$ equation.

## 2 Preliminaries

Let us recall some notations in $\bar{\partial}$-Neumann problem. The solution of the $\bar{\partial}$-Neumann problem will be used in the proof of extension theorem in section 4. We refer the readers to [5] for details. Let $\Omega$ be a domain in $C^n$ with smooth boundary $\partial \Omega$. Let $\partial^*$ and $\vartheta^*$ be the Hilbert space adjoints.
of $\bar{\partial}$ and $\bar{\partial}$ respectively in $L^2(\Omega)$, where $\bar{\partial}$ is the formal adjoint of $\bar{\partial}$. Let us set

$$
D^{p,q} = \text{Dom}(\bar{\partial}^* \cap \wedge^{p+q}(\bar{\Omega})) \quad \text{and} \quad \wedge^{p,q} = \text{Dom}(\partial^* \cap \wedge^{p,q}(\bar{\Omega}))
$$

where $\Box_{p,q}$ is the complex Laplacian for $(p,q)$-forms. Then it follows that $\bar{\partial}^* = \partial$ on $D^{p,q}$ and $\partial^* = \bar{\partial}$ on $\wedge^{p,q}$. The Hodge star operator $\ast : \wedge^{p,q}(\Omega) \to \wedge^{n-p,n-q}(\Omega)$ is defined by the equation $\psi \wedge \ast \phi = \langle \psi, \phi \rangle dV$ where $dV$ is the volume form on $\Omega$. The following results are proved in [5].

$$
\partial = -\ast \partial \ast, \quad \text{and} \quad \wedge^{p,q} = \ast \wedge^{n-p,n-q}.
$$

We say that $\Omega$ satisfies the condition $Z(q)$ if the Levi form has at least $n - q$ positive eigenvalues or at least $q + 1$ negative eigenvalues at each point of $b\Omega$. For the solvability of $\bar{\partial}$-equation, we have the following well known theorem [7].

**Proposition 2.1.** Let $\Omega$ be a smooth bounded domain in $\mathbb{C}^n$. If $\Omega$ has the property $Z(q)$, $q > 0$, and if $\alpha \in \Lambda^{P,q}(\bar{\Omega})$, then there exists $\psi \in D^{p,q+1}$ such that $\bar{\partial}\psi = \alpha$ if and only if $\alpha \in D^{p,q}$, $\partial \alpha = 0$ and $\alpha$ is orthogonal to $\mathcal{H}^{p,q}$.

**Lemma 2.2.** Let $\Omega$ be a smooth bounded domain in $\mathbb{C}^n$. If $\Omega$ satisfies condition $Z(n - q)$, $1 \leq q < n$, $\mathcal{H}^{n-p,n-q} = \{0\}$, and $\alpha \in \mathcal{C}^{p,q}$ with $\partial\alpha = 0$, then there exists $\phi \in \mathcal{C}^{p,q-1}$ such that $\bar{\partial}\phi = \alpha$.

**Proof.** It follows from $\alpha \in \mathcal{C}^{p,q}$ and $\bar{\partial}\alpha = 0$ that $\ast\bar{\alpha} \in D^{n,p,n-q}$ and $\bar{\partial} \ast \bar{\alpha} = 0$. From assumption, $\ast\bar{\alpha}$ is orthogonal to $\mathcal{H}^{n-p,n-q}$, which is a zero space. By the above proposition, there exists $u \in D^{n-p,n-q+1}$ such that $\bar{\partial}u = \ast\bar{\alpha}$, i.e.,

$$
-\ast \bar{\partial} * u = \ast \bar{\alpha}
$$

So, $\partial(-\ast u) = \bar{\alpha}$, $\bar{\partial}(-\ast \bar{u}) = \alpha$. Let $\phi = -\ast \bar{u}$. We have $\bar{\partial}\phi = \alpha$ and $\phi \in \mathcal{C}^{p,q-1}$, where $1 \leq q < n$. This completes the proof.

We need the following theorem [5], which will be used frequently in the proofs of theorems in section 4.

**Theorem 2.3.** Let $\Omega$ be a smooth bounded domain in $\mathbb{C}^n$. Suppose that $\Omega$ satisfies the condition $Z(n - q - 1)$. Then there is a weak $\bar{\partial}$-closed extension of $\phi \in \Lambda^{p,q}(b\Omega)$ if and only if $\bar{\partial}\phi = 0$ and $\int_{\partial\Omega} \theta \wedge \phi = 0$ for all $\theta \in \mathcal{H}^{n-p,n-q-1}$. Moreover the extension is explicitly given by $\phi' + (-1)^{p+q} \ast \bar{\partial}N\partial \ast \bar{\phi}'$, where $\phi'$ is a smooth extension of $\phi$ into $\bar{\Omega}$.

We state a theorem about smooth dependence of solutions of $\bar{\partial}$-Neumann problem with respect to a parameter. Let $\bar{\Omega}$ be a compact manifold of real dimension $2n$ with smooth boundary. Let $\{\mathcal{L}_t, \langle , \rangle_t\}$ be a smooth family of Hermitian structures on $\bar{\Omega}$, where $t = (t_1, \cdots, t_m)$ is a parameter. Suppose $\Omega$ is strongly pseudoconvex with respect to any complex structure $\mathcal{L}_t$ and let $\Lambda^{p,q}_t$ denote the space of smooth $(p,q)$-forms over domain $\Omega$ with respect to the complex structure $\mathcal{L}_t$. Let $\{\alpha_t : \alpha_t \in \Lambda^{p,q}_t\}$ be a smooth family of $(p,q)$-forms. By standard results of $\bar{\partial}$-Neumann problem, there exists a smooth solution $\phi_t \in \Lambda^{p,q}_t$ for the equation $(\Box_t + I)\phi_t = \alpha_t$, where $\Box_t = \bar{\partial}_t \partial_t + \partial_t^{*} \partial_t$ is the Laplacian operator in the Hermitian space $\{\Omega, \mathcal{L}_t, \langle , \rangle_t\}$. It is natural to ask if $\phi_t$ depends on $t$ smoothly.

Let $H_s(W)$ denote the Sobolev space of order $s$ on $W$. With a minor change of the proofs in [4], we have the following theorem.
Theorem 2.4. Let \( \{ \Omega_t \}_{t \in I} \) be smooth family of strongly pseudoconvex domains in \( C^{k_1} \) and let \( N_t \) be the Neumann operator on \( \Omega_t \) for each \( t \in I \), where \( I \) is an open subset of \( C^{k_2} \). Then \( N_t \) depends smoothly on \( t \in I \), i.e., if \( \{ \alpha_t = \alpha(\cdot, t) \}_{t \in I} \) is a smooth family of smooth forms such that \( \alpha_t \in \text{Dom}(N_t) \) for each \( t \in I \), then \( u(z, t) := N_t \alpha_t(z) \) is a smooth form on \( \bigcup_{t \in I} \Omega_t \), which is an open subset of \( C^{k_1 + k_2} \), and satisfies the estimates

\[
\| u \|_{m(\bigcup_{t \in I} \Omega_t)} \leq C_m \| \alpha \|_{3m(\bigcup_{t \in I} \Omega_t)},
\]

provided \( \alpha \in H_{3m}(\bigcup_{t \in I} \Omega_t) \).

We will use this estimate to remove the bad set \( B \) appeared in Xu’s result [8].

3 Local smooth decomposition of domains

Let \( M \) be a real hypersurface in \( C^n \) with smooth defining function \( \rho \) and assume that the Levi-form at \( p \in M \) has \( k \) positive eigenvalues, \( 2 \leq k \leq n - 1 \). In this section we want to find an open neighborhood \( U \) of \( p \in C^n \) such that the set \( U^- = U \cap \{ z : \rho(z) \leq 0 \} \) is decomposed into a smooth family of strongly pseudoconvex domains in \( C^k \). By standard method of holomorphic coordinate changes, we have the following proposition:

Proposition 3.1. Let \( M \) be a smooth hypersurface with smooth defining function \( \rho \), and assume that the Levi form at \( p \in M \) has at least \( k \) positive eigenvalues, \( 2 \leq k \leq n - 1 \). Then there is a holomorphic change of coordinates, \( z = (z_1, \ldots, z_n) \), \( \rho(p) = 0 \), so that, in terms of \( z \) coordinates, the Taylor expansion of \( \rho \) near \( p \) can be written as:

\[
\rho(z) = \Re z_n + \sum_{i=1}^{k} |z_i|^2 + \sum_{i=k+1}^{n-1} \lambda_i |z_i|^2 + O(|z|^3),
\]

where each \( \lambda_i \) is a real number and \( O(|z|^3) \) is the remainder whose first and second derivatives vanish at \( p \).

Using these coordinates and implicit function theorem, we decompose the set \( \{ z : \rho(z) < 0 \} \) into a smooth family of strongly pseudoconvex domains in \( C^k \) near \( 0 \). Let \( z_i = x_{2i-1} + \sqrt{-1}x_{2i} \), \( i = 1, \ldots, k \), and \( z' = (z_1, \ldots, z_k) \), \( z'' = (z_{k+1}, \ldots, z_n) \). For each fixed \( z'' \), we consider the minimum of \( \rho(z', z'') \) over \( z' \). Since \( k < n \), it follows from (3.1) that \( \partial \rho(0, 0)/\partial x_i = 0 \), \( i = 1, \ldots, 2k \), and

\[
\left( \frac{\partial^2 \rho(0, 0)}{\partial x_i \partial x_j} \right)_{1 \leq i, j \leq 2k} = 2I_{2k \times 2k}.
\]

Applying the implicit function theorem to the equations \( \partial \rho(z', z'')/\partial x_i = 0 \), \( 1 \leq i \leq 2k \), there are \( \epsilon_1 > 0 \) and \( \epsilon_2 > 0 \) such that for any \( z'' \) with \( |z''| < \epsilon_2 \), there exists a unique vector \( z'(z'') \), \( |z'(z'')| < \epsilon_1 \) such that \( \partial \rho(z'(z''), z'')/\partial x_i = 0 \), \( 1 \leq i \leq 2k \), and

\[
\left( \frac{\partial^2 \rho(z'(z''), z'')}{\partial x_i \partial x_j} \right)_{1 \leq i, j \leq 2k} \approx 2I_{2k \times 2k}
\]

by (3.2). Therefore \( \rho(z', z'') \) achieves minimum at \( z'(z'') \). Set \( P_{\epsilon_1, \epsilon_2} = \{ z = (z', z'') \in C^n : |z'| < \epsilon_1, \ |z''| < \epsilon_2 \} \) and set \( s(z'') = \rho(z'(z''), z'') \). Then it follows that \( \Omega_{z''} = \{ z' \in C^k : \rho(z', z'') < 0 \} \) is not empty when \( s(z'') < 0 \). For those \( (z', z'') \) near \( 0 \) with \( z' \neq z'(z'') \), we note that \( \partial \rho(z'(z''), z'')/\partial x_i \neq 0 \) for some \( 1 \leq i \leq 2k \). Hence it follows that \( \Omega_{z''} \) is a smooth strictly convex region in \( C^k \) when \( s(z'') < 0 \) and \( |z''| < \epsilon_2 \). In summary, we have the following Lemma.
Lemma 3.2. Let $\mathcal{M}$ be a smooth hypersurface with smooth defining function $\rho$, and assume that the Levi form at $p \in \mathcal{M}$ has at least $k$ positive eigenvalues, $2 \leq k \leq n - 1$. Then there is a new holomorphic change of coordinates $z = (z_1, \ldots, z_n)$, $z(p) = 0$, so that in terms of $z$ coordinate, the open set $\{ z \in P_{\varepsilon_1, \varepsilon_2} : \rho(z) < 0 \}$ can be expressed as the union of strongly pseudoconvex domains in $\mathbb{C}^k$, that is,

$$\{ z \in P_{\varepsilon_1, \varepsilon_2} : \rho(z) < 0 \} = \bigcup_{z'' \in I} \Omega_{z''} \times \{ z'' \},$$

(3.3)

where $\Omega_{z''} = \{ z' \in \mathbb{C}^k : \rho(z', z'') < 0 \}$, and $I = \{ z'' \in \mathbb{C}^{n-k} : |z''| < \varepsilon_2, s(z'') < 0 \}$.

Remark 3.3. (1) We denote the boundary of the domain $\Omega_{z''}$ in $\mathbb{C}^k$ by $b\Omega_{z''}$. From the definition of $\Omega_{z''}$, we have

$$\bigcup_{z'' \in I} b\Omega_{z''} \times \{ z'' \} = \mathcal{M} \setminus B,$$

where

$$B = \{(z', z'') : |z'| < \varepsilon_1, |z''| < \varepsilon_2, s(z'') = 0, \rho(z', z'') = 0 \}. \quad (3.4)$$

From the definition of $B$, it follows that $B$ is a $2n - 2k - 1$ real dimension variety where the set $\{ z'' \in \mathbb{C}^k : \rho(z', z'') \leq 0 \}$ is a one point set.

(2) $\{ \Omega_{z''} \}_{z'' \in I}$ is a smooth family of strongly pseudoconvex domains in $\mathbb{C}^k$ with parameter $z'' \in I$.

(3) If we set $U = P_{\varepsilon_1, \varepsilon_2}$, then by definition, $U^- = \{ z \in P_{\varepsilon_1, \varepsilon_2} : \rho(z) \leq 0 \}$ and

$$U^- \setminus B = \bigcup_{z'' \in I} \Omega_{z''} \times \{ z'' \},$$

by (3.3). That is, $U^-$ is decomposed into a smooth family of strongly pseudoconvex domains in $\mathbb{C}^k$.

4 $\bar{\partial}$-closed extension for $(0, r)$-forms

To prove local extension theorem, we use the local decomposition of the set $\{ z \in P_{\varepsilon_1, \varepsilon_2} : \rho(z) < 0 \}$ considered in the previous section with $k = q + 1$. From now on we fix $k = q + 1$. We will use the $\bar{\partial}$-Neumann technique on each strongly pseudoconvex domain $\Omega_{z''}$, and the smooth dependence of the Neumann operator on $z'' \in I$. Set $\mathcal{K} = \{ 1, \ldots, k \} = \{ 1, \ldots, q + 1 \}$, and $\mathcal{K}^c = \{ k + 1, \ldots, n \} = \{ q + 2, \ldots, n \}$. For a smooth function $f$ defined in $\mathbb{C}^n$, we define

$$\bar{\partial}_{\mathcal{K}} f = \sum_{j=1}^{k} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j, \quad \bar{\partial}_{\mathcal{K}^c} f = \sum_{j=k+1}^{n} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j,$$

We can extend this definition for arbitrary smooth forms. We note that $\wedge^p q(\Omega_{z''})$, $D^p q(\Omega_{z''})$ and $\wedge^p q(b\Omega_{z''})$ are defined on $\Omega_{z''} \subset \mathbb{C}^k = \mathbb{C}^{q+1}$, and every summation will be over strictly increasing indices.

Proposition 4.1. Let $\mathcal{M}$ be a real hypersurface in $\mathbb{C}^n$, $n \geq 3$, and suppose that the Levi-form at $p \in \mathcal{M}$ has $(q + 1)$ positive eigenvalues, $1 \leq q \leq n - 2$. Then there is a neighborhood $U$ of
First, let us recall our decompositions:

\[ \partial \alpha_1 = \sum_{J \subseteq K, M \subseteq K^c} H_{JM} d \tilde{z}^J \wedge d \bar{z}^M \]

on \( U^- \setminus \mathcal{M} \) for some smooth functions \( H_{JM} \) on \( U^- \setminus B \).

**Proof.** First, let us recall our decompositions:

\[ U^- = \{ z \in P_{\varepsilon_1, \varepsilon_2} : \rho(z) \leq 0 \}, \]
\[ U^- \setminus \mathcal{M} = \bigcup_{z'' \in I} \Omega_{z''} \times \{ z'' \}, \]
\[ U^- \setminus B = \bigcup_{z'' \in I} \Omega_{z''} \times \{ z'' \}. \]

For a convenience of notations, we denote \( \Omega_{z''} \times \{ z'' \} \) by \( \Omega_{z''} \). Therefore the open set \( \bigcup_{z'' \in I} \Omega_{z''} \times \{ z'' \} \subset \mathbb{C}^n \) will be denoted by \( \bigcup_{z'' \in I} \Omega_{z''} \).

**Case I** \( 1 \leq r \leq k-2 \). By Lemma 3.2 and Remark 3.3, there are new holomorphic coordinates \( z = (z_1, \ldots, z_n) \), \( z(p) = 0 \), so that in terms of \( z \) coordinate, the set \( U^- := \{ z \in P_{\varepsilon_1, \varepsilon_2} : \rho(z) \leq 0 \} \) can be expressed as the union of strongly pseudoconvex domains in \( \mathbb{C}^k \). First, we show that there is a \( (\partial \bar{\kappa})_b \)-closed form \( \alpha_1 \) extracted from \( \alpha \). Let \( E\alpha \) be a smooth extension of \( \alpha \) on \( U^- \). We decompose \( E\alpha \) into two parts

\[ E\alpha = \sum_{J \subseteq K} a_J d \tilde{z}^J + \sum_{J \not\subseteq K} a_J d \tilde{z}^J := \alpha^1 + \alpha^2 \] in \( U^- \).

Note that

\[ \alpha^1 |_{\Omega_{z''}}(z') = \sum_{J \subseteq K} a_J(z', z'') d \tilde{z}^J \in \Lambda^{0,r}(\Omega_{z''}) \]

for each fixed \( z'' \). From the definition of \( D^{0,r}(\Omega_{z''}) \) and \( C^{0,r}(\Omega_{z''}) \), we can decompose \( \alpha^1 \) into two parts; \( \alpha^1 = \alpha' + \alpha'' \), where \( \alpha' |_{\Omega_{z''}} \in D^{0,r}(\Omega_{z''}) \) and \( \alpha'' |_{\Omega_{z''}} \in C^{0,r}(\Omega_{z''}) \). Let us write

\[ \bar{\partial} \alpha' \wedge \bar{\partial} \rho = (\bar{\partial} \kappa + \bar{\partial} \kappa')(\alpha^1 + \alpha^2) \wedge (\bar{\partial} \kappa + \bar{\partial} \kappa') \rho \]
\[ = \sum_{|J|=q+2} f_J d \tilde{z}^J = \sum_{J \subseteq K} f_J d \tilde{z}^J + \sum_{J \not\subseteq K} f_J d \tilde{z}^J. \]

(4.1)

Since \( \bar{\partial} \alpha = 0 \) on \( \mathcal{M} \cap U^- \), it follows that \( \bar{\partial} \alpha' \wedge \bar{\partial} \rho = 0 \) on \( \mathcal{M} \cap U^- \). Thus one obtains from (4.1) that

\[ \bar{\partial} \kappa \alpha^1 \wedge \bar{\partial} \kappa \rho = \sum_{J \subseteq K} f_J d \tilde{z}^J = 0 \] on \( \mathcal{M} \cap U^- \).

(4.2)

This shows that \( (\bar{\partial} \kappa)_b(\alpha' |_{\bar{\partial} \kappa \Omega_{z''}}) = 0 \) on \( \bar{\partial} \kappa \Omega_{z''} \) for each \( z'' \in I \).

Next, we find a weak \( \bar{\partial} \kappa \)-closed extension \( \beta_{z''} \) of \( \alpha' |_{\bar{\partial} \kappa \Omega_{z''}} \) on \( \Omega_{z''} \) which depends smoothly on \( z'' \in I \). Since \( \Omega_{z''} \) is a strongly pseudoconvex domain in \( \mathbb{C}^k \) and \( (\bar{\partial} \kappa)_b(\alpha' |_{\bar{\partial} \kappa \Omega_{z''}}) = 0 \) for each
$z'' \in I$, it follows from Theorem 5.3.1 in [5] that there exists a $\bar{\partial}_K$-closed extension of $\alpha'|_{\Omega_{z''}}$ on $\Omega_{z''}$ for each $z'' \in I$. That is, there exists $\beta_{z''} \in \wedge^0 r(\Omega_{z''})$ such that

$$\bar{\partial}_K \beta_{z''} = 0 \text{ in } \Omega_{z''}, \quad \text{and} \quad (\beta_{z''} - \alpha'|_{\Omega_{z''}}) \wedge \bar{\partial}_K \rho = 0 \text{ on } b\Omega_{z''}. \quad (4.3)$$

By Theorem 2.3, $\beta_{z''}$ is explicitly given by $\beta_{z''} = \alpha'|_{\Omega_{z''}} + (-1)^r \ast_K \bar{\partial}_K \bar{N}_{z''} \bar{\partial}_K \ast_K \bar{\alpha}'|_{\Omega_{z''}}$, where $N_{z''}$ is the $\bar{\partial}$-Neumann operator on $\Omega_{z''}$, $\bar{\partial}_K$ is the formal adjoint of $\partial_K$ on $\Omega_{z''}$, and $\ast_K$ is the hodge star operator in $\mathbb{C}^k$. By Theorem 2.4, we note that $N_{z''}$ depends smoothly on $z''$. Since the other operators depend smoothly on $z''$, it follows that $\beta_{z''}$ depends smoothly on $z'' \in I$. We define a smooth $(0, q)$-form $\beta$ on $U^- \setminus B$ by

$$\beta(z', z'') = \beta_{z''}(z'). \quad (4.4)$$

Finally, we construct a $(0, q)$-form $\alpha_1$ such that $(\alpha_1 - \alpha) \wedge \bar{\partial}_K \rho = 0$ on $\mathcal{M} \cap U^-$ and $\bar{\partial} \alpha_1$ has no $d\bar{z}^L$ term for every $L$ with $|L| = q + 1$ and $L \subset \mathcal{K} = \{1, \ldots, k\}$. From (4.3), it follows that

$$(\beta_{z''} - \alpha^1) \wedge \bar{\partial}_K \rho = (\beta_{z''} - \alpha') \wedge \bar{\partial}_K \rho - \alpha'' \wedge \bar{\partial}_K \rho = 0$$

because $\alpha''|_{\Omega_{z''}} \in \mathcal{C}^{0, \beta}(\Omega_{z''})$ for each $z'' \in I$. Thus we can write $\beta - \alpha^1 = \nu \wedge \bar{\partial}_K \rho + \rho \theta$ for some smooth $(0, q - 1)$-form $\nu$ and $(0, q)$-form $\theta$ on $U^- \setminus B$. Set $\tilde{\beta} = \beta + \nu \wedge \bar{\partial}_K \rho + \alpha^2$ on $U^- \setminus B$. Then

$$\tilde{\beta} - \alpha = \left((\nu \wedge \bar{\partial}_K \rho + \alpha^2) - (\alpha^1 + \alpha^2)\right) = (\nu \wedge \bar{\partial}_K \rho + \rho \theta)$$

on $U^- \setminus B$. This implies that $(\tilde{\beta} - \alpha) \wedge \bar{\partial}_K \rho = 0$ on $(\mathcal{M} \cap U^-) \setminus B$. Moreover we have

$$\bar{\partial} \tilde{\beta} = \bar{\partial}(\tilde{\beta} - \alpha^1 + \nu \wedge \bar{\partial}_K \rho)$$

$$= \left(\bar{\partial}(\nu \wedge \bar{\partial}_K \rho + \alpha^2) + \nu \wedge \bar{\partial}_K \rho \right)$$

$$= \nu \wedge \bar{\partial}_K \rho + \rho \theta$$

on $U^- \setminus \mathcal{M}$ because $\bar{\partial}_K \beta = 0$ on $U^- \setminus \mathcal{M}$ by (4.3). Since every term in (4.5) has either $\bar{\partial}_K \nu$ or $\alpha^2$, we can write

$$\bar{\partial} \tilde{\beta} = \sum_{J \subset K, M \subset K^c, |M| \geq 1, |J| + |M| = q + 1} H_{JM} d\bar{z}^J \wedge d\bar{z}^M. \quad (4.6)$$

on $U^- \setminus \mathcal{M}$ for some smooth functions $H_{JM}$ on $U^- \setminus B$. If we set $\alpha_1 = \tilde{\beta}$, then this proves for the case that $1 \leq q \leq k - 2$.

**Case II: $r = k - 1 = q$.**

When $r = k - 1 = q$, we can show the same result but in different method. By Lemma 2.3.1 of [8], there exists $E \alpha \in \wedge^{0, k-1}(U^-)$ such that $\bar{\partial} E \alpha = O(|\rho|^{\infty})$ and $E \alpha = \alpha$ on $\mathcal{M}$. As in Step 1, we decompose $E \alpha$ into two parts

$$E \alpha = \sum_{J \subset K} a_J d\bar{z}^J + \sum_{J \not\subset K} a_J d\bar{z}^J := \alpha^1 + \alpha^2 \text{ in } U^- \setminus B,$$
and then decompose $\alpha^1$ into two parts: $\alpha^1 = \alpha' + \alpha''$, where $\alpha'|_{\Omega_{z''}} \in D^{0,2}(\Omega_{z''})$ and $\alpha''|_{\Omega_{z''}} \in C^{0,q}(\Omega_{z''})$ for each $z'' \in I$.

Note that $\alpha'|_{\Omega_{z''}} \in B^{0,k-1}(b_{\Omega_{z''}})$ for each $z'' \in I$. By Theorem 2.3, there exists a $\partial K$-closed extension of $\alpha'$ into $\Omega_{z''}$ if and only if

$$\int_{b_{\Omega_{z''}}} \alpha' \wedge \psi = 0$$

for every $\psi \in \Lambda^{k,0}(\Omega_{z''}) \cap Ker(\partial K)$

(4.7)

because $\Omega_{z''}$ is a bounded strongly pseudoconvex domain in $C^k$. We will show (4.7) for each $z'' \in I$. Let $z_0'' \in I$ be fixed. Since $I = \{z'' \in C^{n-k} : |z''| < \varepsilon_2, s(z'') < 0\}$, $I$ is an open subset of $C^{n-k}$. Thus there exists $\varepsilon > 0$ such that $\{z'' \in C^{n-k} : |z'' - z_0''| < \varepsilon\} \subset I$. Let $\phi_{z_0''} : C^n \rightarrow R$ be a smooth function satisfying

$$\phi_{z_0''}(z', z'') = \left\{ \begin{array}{ll}
1 & \text{if } |z'' - z_0''| < \varepsilon/4 \\
0 & \text{if } |z'' - z_0''| > \varepsilon/2,
\end{array} \right.$$

where $\phi_{z_0''}(z', z'')$ does not depend on $z'$, i.e. $\phi_{z_0''}(\cdot , z'')$ is a constant function on each $C^k \times \{z''\}$.

Set $G(z', z'') = \bar{\partial}(\phi_{z_0''}(z', z'')\alpha^1(\cdot , z'')) = \sum_{|j|=k} G_{j} d\bar{z}^j$. If we write terms containing $d\bar{z}^K \wedge d\bar{z}_l = (d\bar{z}_1 \cdots \wedge d\bar{z}_k) \wedge d\bar{z}_l$ with $l \in K^c$ in the expression of $\bar{\partial}G$, we obtain that

$$\sum_{l=k+1}^n (-1)^k \frac{\partial G_{K}}{\partial \bar{z}_l} + \sum_{j=1}^k (-1)^j \frac{\partial G_{(K\setminus\{j\})l}}{\partial \bar{z}_j} d\bar{z}^K \wedge d\bar{z}_l = 0,$$

where $\langle (K \setminus \{j\})l \rangle$ denotes the $k$-tuple obtained by reordering the set $(K \setminus \{j\}) \cup \{l\}$. This implies that for each $l \in K^c$,

$$\frac{\partial G_{K}}{\partial \bar{z}_l} = \sum_{j=1}^k (-1)^{j+k} \frac{\partial G_{(K\setminus\{j\})l}}{\partial \bar{z}_j}.$$

(4.8)

For a fixed $\psi \in \Lambda^{k,0}(\Omega_{z''}) \cap Ker(\partial K)$, define $F : C^{n-k} \rightarrow C$ by

$$F(z'') = \int_{\Omega_0} \bar{\partial}_K(\phi_{z_0''}(\cdot , z'')\alpha^1(\cdot , 0)) \wedge \psi(\cdot).$$

It is easy to check that $\bar{\partial}_K = O(|\rho|^{\infty})$ implies that $\bar{\partial}_K \alpha^1 = O(|\rho|^{\infty})$. In the expression of $G$, the $d\bar{z}^K$ term is given by

$$G_{K}(\cdot , z'')d\bar{z}^K = \bar{\partial}_K(\phi_{z_0''}(\cdot , z'')\alpha^1(\cdot , z_0'')).$$

Since $\phi_{z_0''}(\cdot , z_0'')$ does not depend on $z''$, we have

$$G_{K}(\cdot , z'')d\bar{z}^K = \phi_{z_0''}(\cdot , z'')\bar{\partial}_K \alpha^1(\cdot , z_0'')$$

for each $z''$. Hence $G_K = O(|\rho|^{\infty})$ because $\bar{\partial}_K \alpha^1 = O(|\rho|^{\infty})$. Since $\psi \in \Lambda^{k,0}(\Omega_{z''}) \cap Ker(\partial K)$, $\psi$ can be expressed as $\psi = \psi_K d\bar{z}^K$, where $\psi_K$ is a holomorphic function in $\Omega_{z''}$. By (4.8) and by
Proposition 4.2. Let \( M \) be a real hypersurface in \( C^n, n \geq 3 \), and suppose that the Levi-form at \( p \in M \) has \((q + 1)\) positive eigenvalues, \( 1 \leq q \leq n - 2 \). Then there is a neighborhood \( U \) of \( p \) such that for any \( \alpha \in \wedge^0\gamma(M \cap U) \), \( 1 \leq r \leq q \), satisfying \( \partial_h \alpha = 0 \) on \( M \cap U \), there exists
\( \alpha_j \in \wedge^{0,q}(U^c \setminus B) \) for each \( 1 \leq j \leq q + 1 \), such that \((\alpha_j - \alpha) \wedge \partial \rho = 0 \) on \( \mathcal{M} \setminus B \) and \( \partial \alpha_j \) can be written as

\[
\partial \alpha_j = \sum_{J \subset K, M \subset K^c} H_{JM} d\bar{z}^J \wedge d\bar{z}^M
\]

on \( U^c \setminus \mathcal{M} \) for some smooth functions \( H_{JM} \) on \( U^c \setminus B \).

**Proof.** We will use induction on \( j \). Note that for \( j = 1 \) the statement is true by Proposition 4.1. We assume that the statement is true for \( j = l \) and then we will show that it is true for \( j = l + 1 \). By assumption there exists \( \alpha_l \in \wedge^{0,q}(U^c \setminus B) \) such that \((\alpha_l - \alpha) \wedge \partial \rho = 0 \) on \( \mathcal{M} \setminus B \) and

\[
G := \partial \alpha_l = \sum_{I \subset K, M \subset K^c} G_{IM} d\bar{z}^I \wedge d\bar{z}^M
\]

for some smooth functions \( G_{IM} \in C^\infty(U^c \setminus B) \). Note that

\[
\partial G = \sum_{j=1}^n \sum_{I \subset K, M \subset K^c \atop |M| = q + 1} \frac{\partial G_{IM}}{\partial z_j} d\bar{z}_j \wedge d\bar{z}^I \wedge d\bar{z}^M = 0 \tag{4.9}
\]

on \( U^c \setminus \mathcal{M} \). We can decompose \( \partial G \) into two parts

\[
\partial G = \sum_{J \subset K, M \subset K^c \atop |M| = l, |J| = q-l+2} F_{JM} d\bar{z}^J \wedge d\bar{z}^M + \sum_{J \subset K, M \subset K^c \atop |M| = l+1, |J| = q+1} F_{JM} d\bar{z}^J \wedge d\bar{z}^M =: A + B.
\]

From (4.9), we have

\[
A = \sum_{j=1}^k \sum_{I \subset K, M \subset K^c \atop |M| = l, |I| = q-l+1} \frac{\partial G_{IM}}{\partial z_j} d\bar{z}_j \wedge d\bar{z}^I \wedge d\bar{z}^M
\]

\[
= \sum_{I \subset K, M \subset K^c} \sum_{j \in K, j \subset J \subset K \atop |I| = q-l+1} \epsilon_{J}^I \frac{\partial G_{IM}}{\partial z_j} d\bar{z}_j \wedge d\bar{z}^I \wedge d\bar{z}^M = 0
\]

on \( U^c \setminus \mathcal{M} \), where \( \epsilon_{J}^I \) is the sign of the permutation taking \( jI \) to \( J \), which equals to 0 if \( \{j\} \cup I \neq J \). Hence we have

\[
\sum_{j \in K, I \subset K \atop |I| = q-l+1} \epsilon_{J}^I \frac{\partial G_{IM}}{\partial z_j} = 0 \tag{4.10}
\]

on \( U^c \setminus \mathcal{M} \) for each index \( J \subset K \) and \( M \subset K^c \) with \( |J| = q-l+2 \) and \( |M| = l \).

We define a \((0,q-l+1)\)-form \( \Phi^M \) on \( U^c \setminus B \) by

\[
\Phi^M = \sum_{I \subset K \atop |I| = q-l+1} G_{IM} d\bar{z}^I. \tag{4.11}
\]
Then it follows from (4.10) that
\[ \bar{\partial}_K \Phi^M \bigg|_{M \cup |z''|} = \sum_{j \in K, |I| = q - l + 1, I \subset K} \frac{\partial G_{IM}}{\partial \bar{z}_j} d\bar{z}_j \wedge d\bar{z}_I \]

\[ = \sum_{j \in K, I \subset K} c^{I}_j \frac{\partial G_{IM}}{\partial \bar{z}_j} d\bar{z}_I = 0 \]
on \(U^\perp \setminus M\). Hence \(\Phi^M |_{\Omega''} \) is \(\bar{\partial}_K\)-closed and smooth up to boundary \(b\Omega''\) in \(\Omega''\) for each \(z'' \in I\).

We will show that \(\Phi^M \wedge \bar{\partial}_K \rho = 0 \) on \(M \setminus B\). For a simplicity, we set \(M \cap \Omega'' = M\) because of the local nature of our problem. Since \(\bar{\partial}_\alpha = 0\) on \(M \setminus B\), it follows that \(\bar{\partial}_\alpha \wedge \bar{\partial}_\rho = 0\) on \(M \setminus B\). Also we have \(\bar{\partial}(\alpha_1 - \alpha) \wedge \bar{\partial}_\rho = 0\) on \(M \setminus B\) because \((\alpha_1 - \alpha) \wedge \bar{\partial}_\rho = 0\) on \(M \setminus B\). Therefore one obtains that \(G \wedge \bar{\partial}_\rho = \bar{\partial}_\alpha \wedge \bar{\partial}_\rho = 0\) on \(M \setminus B\). In the expression of \(G \wedge \bar{\partial}_\rho\), if we write terms involving \(d\bar{z}_J \wedge d\bar{z}_M\) with \(J \subset K, M \subset \Omega'', |M| = 1\), we obtain that

\[ \sum_{M \subset K} \left( \sum_{I \subset K} \sum_{j \in K, J \subset K} (-1)^{I} c^{j}_{I} G_{IM} \frac{\partial \rho}{\partial \bar{z}_j} d\bar{z}_I \right) \wedge d\bar{z}_M = 0 \quad (4.12) \]
on \(U^\perp \setminus B\). Therefore it follows from (4.12) that

\[ \Phi^M \wedge \bar{\partial}_K \rho = \sum_{|I| = q - l + 1} G_{IM} d\bar{z}_I \wedge \left( \sum_{j=1}^{k} \frac{\partial \rho}{\partial \bar{z}_j} d\bar{z}_j \right) \]

\[ = \sum_{|I| = q - l + 1} \sum_{j \in K, I \subset K} c^{j}_{I} G_{IM} \frac{\partial \rho}{\partial \bar{z}_j} d\bar{z}_I = 0 \]
on \(M \setminus B\). This implies that \(\Phi^M |_{\Omega''} \in C^{0, q - I + 1}(\Omega''\setminus I)\) for each parameter \(z'' \in I\). Since \(\Phi^M |_{\Omega''} \) is \(\bar{\partial}_K\)-closed and \(\Omega''\) is a bounded strongly pseudoconvex domain, there exists \(u_M(\cdot, z'') \in C^{0, q - l}(\Omega''\setminus I)\) such that \(\bar{\partial}_K u_M(\cdot, z'') = \Phi^M |_{\Omega''}\) on \(\Omega''\) by Lemma 2.2. We need to prove that \(u_M(\cdot, z'')\) depends smoothly on the parameter \(z'' \in I\). In view of the proof of Lemma 2.2, it suffices to show that the Neumann operator \(N_{z''}\) depends smoothly on \(z''\), which follows from Theorem 2.4.

Since \(u_M \wedge \bar{\partial}_K \rho = 0\) \((u_M \in C^{0, q - I}(\Omega''\setminus I))\) on \(M \setminus B\), we can write

\[ u_M = \phi_M \wedge \bar{\partial}_K \rho + \rho \psi_M \]

for some \(\phi_M \in \wedge^{0, q - l - 1}(U^\perp \setminus B)\) and \(\psi_M \in \wedge^{0, q - l}(U^\perp \setminus B)\). Set \(\tilde{u}_M = u_M + \phi_M \wedge \bar{\partial}_K \rho\) on \(U^\perp \setminus B\). Then

\[ \tilde{u}_M \wedge \bar{\partial}_\rho = (\phi_M \wedge \bar{\partial}_\rho + r \psi_M) \wedge \bar{\partial}_\rho = 0 \quad (4.13) \]
on $\mathcal{M} \setminus B$. Set $w = \sum_{M \subset \mathcal{K}^e, |M|=l} \tilde{u}_M \wedge d\tilde{z}^M$. Then it follows form (4.11) that

$$
\tilde{\partial} w = \tilde{\partial}_\mathcal{K} \left( \sum_{M \subset \mathcal{K}^e, |M|=l} \tilde{u}_M \wedge d\tilde{z}^M \right) + \tilde{\partial}_\mathcal{K}^e \left( \sum_{M \subset \mathcal{K}^e, |M|=l} \tilde{u}_M \wedge d\tilde{z}^M \right)
= \sum_{M \subset \mathcal{K}^e, |M|=l} \tilde{\partial}_\mathcal{K} u_M \wedge d\tilde{z}^M + \sum_{M \subset \mathcal{K}^e, |M|=l} \tilde{\partial}_\mathcal{K} (\phi_M \wedge \tilde{\partial}_\mathcal{K}^e \rho) \wedge d\tilde{z}^M + \tilde{\partial}_\mathcal{K}^e \left( \sum_{M \subset \mathcal{K}^e, |M|=l} \tilde{u}_M \wedge d\tilde{z}^M \right)
= \sum_{M \subset \mathcal{K}^e, |M|=l} \Phi^M \wedge d\tilde{z}^M + \sum_{M \subset \mathcal{K}^e, |M|=l} \left( \tilde{\partial}_\mathcal{K} (\phi_M \wedge \tilde{\partial}_\mathcal{K}^e \rho) + \tilde{\partial}_\mathcal{K}^e \tilde{u}_M \right) \wedge d\tilde{z}^M
= \sum_{J \subset \mathcal{K}, M \subset \mathcal{K}^e} G_J M \tilde{z}^I \wedge d\tilde{z}^M + \sum_{M \subset \mathcal{K}^e, |M|=l} \left( \tilde{\partial}_\mathcal{K} (\phi_M \wedge \tilde{\partial}_\mathcal{K}^e \rho) + \tilde{\partial}_\mathcal{K}^e \tilde{u}_M \right) \wedge d\tilde{z}^M.
$$

(4.14)

Set $\alpha_{l+1} = \alpha_l - w$ on $U^- \setminus B$. On $\mathcal{M} \setminus B$, it follows from (4.13) that

$$
\alpha_{l+1} - \alpha_l = -w \wedge \tilde{\partial} \rho = -w \wedge \tilde{\partial} \rho = 0.
$$

(4.15)

and hence one obtains, by (4.15), that

$$(\alpha_{l+1} - \alpha_l) \wedge \tilde{\partial} \rho = -w \wedge \tilde{\partial} \rho = 0.
$$

(4.16)

By induction hypothesis, $\tilde{\partial} \alpha_l$ can be written as

$$
\tilde{\partial} \alpha_l = \sum_{J \subset \mathcal{K}, M \subset \mathcal{K}^e} G_{JM} \tilde{z}^I \wedge d\tilde{z}^M
= \sum_{J \subset \mathcal{K}, M \subset \mathcal{K}^e} G_{JM} \tilde{z}^I \wedge d\tilde{z}^M + \sum_{J \subset \mathcal{K}, M \subset \mathcal{K}^e} G_{JM} \tilde{z}^I \wedge d\tilde{z}^M.
$$

(4.17)

Combining (4.14) and (4.17), we have

$$
\tilde{\partial} \alpha_{l+1} = \tilde{\partial} \alpha_l - \tilde{\partial} w
= \left( \sum_{J \subset \mathcal{K}, M \subset \mathcal{K}^e} G_{JM} \tilde{z}^I \wedge d\tilde{z}^M + \sum_{J \subset \mathcal{K}, M \subset \mathcal{K}^e} G_{JM} \tilde{z}^I \wedge d\tilde{z}^M \right)
- \left( \sum_{J \subset \mathcal{K}, M \subset \mathcal{K}^e} G_{JM} \tilde{z}^I \wedge d\tilde{z}^M + \sum_{J \subset \mathcal{K}, M \subset \mathcal{K}^e} G_{JM} \tilde{z}^I \wedge d\tilde{z}^M \right)
= \sum_{J \subset \mathcal{K}, M \subset \mathcal{K}^e} G_{JM} \tilde{z}^I \wedge d\tilde{z}^M
- \sum_{M \subset \mathcal{K}^e, |M|=l} \left( \tilde{\partial}_\mathcal{K} (\phi_M \wedge \tilde{\partial}_\mathcal{K}^e \rho) + \tilde{\partial}_\mathcal{K}^e \tilde{u}_M \right) \wedge d\tilde{z}^M
= \sum_{I \subset \mathcal{K}, M \subset \mathcal{K}^e} H_I \tilde{z}^I \wedge d\tilde{z}^M
$$

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for some smooth functions $H_{1M}$ on $U^- \setminus B$. This completes the proof. \qed

Now we prove local extension problem with some bad set $B$:

**Theorem 4.4.** (Weak $\bar{\partial}$-Closed Extension Theorem) Let $M$ be a real hypersurface in $\mathbb{C}^n, n \geq 3$, and suppose that the Levi-form at $p \in M$ has $(q+1)$ positive eigenvalues, $1 \leq q \leq n-2$. Then there is a neighborhood $U$ of $p$ such that for any $\alpha \in \Lambda^{0,r}(\mathcal{M} \cap U), 1 \leq r \leq q$, satisfying $\bar{\partial}_b \alpha = 0$ on $\mathcal{M}$, there exists $\tilde{\alpha} \in \Lambda^{0,r}(U^- \setminus B)$ such that $\bar{\partial} \tilde{\alpha} = 0$ in $U^- \setminus \mathcal{M}$ and $(\tilde{\alpha} - \alpha) \wedge \bar{\partial} \rho = 0$ on $(\mathcal{M} \cap U) \setminus B$.

**Proof.** By Proposition 4.2, there exists $\alpha_{q+1} \in \Lambda^{0,q}(U^- \setminus B)$ such that $(\alpha_{q+1} - \alpha) \wedge \bar{\partial} \rho = 0$ on $\mathcal{M} \setminus B$ and

$$\bar{\partial} \alpha_{q+1} = \sum_{M \subset \mathcal{K}^r \{M = q+1 \}} H_M dz^M \tag{4.18}$$

for some smooth functions $H_M$ on $U^- \setminus B$. If we consider the coefficients of $dz_i \wedge dz^M$ of $\bar{\partial}^2 \alpha_{q+1}$ for $1 \leq i \leq k$, it follows that

$$\left. \frac{\partial H_M}{\partial z_i} \right| = 0.$$

Hence $H_M$ is a holomorphic function on $\Omega_{z''}$. From (4.16), one obtains that $\bar{\partial}(\alpha_{q+1} - \alpha) \wedge \bar{\partial} \rho = 0$ on $\mathcal{M} \setminus B$, and hence $\bar{\partial} \alpha_{q+1} \wedge \bar{\partial} \rho = 0$ on $\mathcal{M} \setminus B$ because $\bar{\partial} \alpha \wedge \bar{\partial} \rho = 0$ on $\mathcal{M} \setminus B$. By considering the coefficients of $dz_i \wedge dz^M$ of $\bar{\partial} \alpha_{q+1} \wedge \bar{\partial} \rho$, it follows from (4.18) that, for every $i$ with $1 \leq i \leq k$,

$$\left. \frac{\partial \rho}{\partial z_i} H_M \right| = 0$$

on $\mathcal{M} \setminus B$. Since $\bar{\partial}_b \rho \neq 0$ on $b \Omega_{z''}$, at least one of $\frac{\partial \rho}{\partial z_i}$ for $1 \leq i \leq k$ is not equal to zero and hence $H_M = 0$ on $b \Omega_{z''}$. Since $H_M$ is a holomorphic function on $\Omega_{z''}$, it follows that $H_M \equiv 0$ and hence $\bar{\partial} \alpha_{q+1} = 0$ on $U^- \setminus B$. If we set $\tilde{\alpha} = \alpha_{q+1}$, then the theorem is proved. \qed

Now we prove our main theorem, that is, the extension problem without the bad set $B$.

**Theorem 4.4.** (Weak $\bar{\partial}$-Closed Extension Theorem) Let $M$ be a real hypersurface in $\mathbb{C}^n, n \geq 3$, and suppose that the Levi-form at $p \in M$ has $(q+1)$ positive eigenvalues, $1 \leq q \leq n-2$. Then there is a neighborhood $U$ of $p$ such that for any $\alpha \in \Lambda^{0,r}(\mathcal{M} \cap U), 1 \leq r \leq q$, satisfying $\bar{\partial}_b \alpha = 0$ on $\mathcal{M}$, there exists $\tilde{\alpha} \in \Lambda^{0,r}(U^-)$ such that $\bar{\partial} \tilde{\alpha} = 0$ in $U^- \setminus \mathcal{M}$ and $(\tilde{\alpha} - \alpha) \wedge \bar{\partial} \rho = 0$ on $\mathcal{M} \cap U$.

**Proof.** Note that $\tilde{\alpha} \in \Lambda^{0,r}(U^- \setminus B)$ by Theorem 4.3. To prove $\tilde{\alpha} \in \Lambda^{0,r}(U^-)$, it suffices to show that $\|\tilde{\alpha}\|_{m(U^- \setminus \mathcal{M})} < \infty$ for each integer $m > 0$. In the process to construct $\tilde{\alpha}$ in Proposition 4.1 and Proposition 4.2, we used the smooth decompositions $\alpha^1, \alpha^2 \in \Lambda^{0,r}(\overline{U^-})$ of the smooth form $E\alpha \in \Lambda^{0,r}(\overline{U^-})$ and the operators $N_{z''}, \bar{\partial}_K, \bar{\partial}_K$, and $\bar{\partial}$. From Theorem 2.4, $N_{z''}$ is a bounded operator from $H_{3m}(\cup_{z'' \in \Omega} \Omega_{z''})$ to $H_{m}(\cup_{z'' \in \Omega} \Omega_{z''})$ for each integer $m > 0$, independent of $z''$. Since $\bar{\partial}_K, \bar{\partial}_K$, and $\bar{\partial}$ are first order differential operators, they are also a bounded operator from $H_{m+1}(\cup_{z'' \in \Omega} \Omega_{z''})$ to $H_m(\cup_{z'' \in \Omega} \Omega_{z''})$. Note that each $\alpha_j$ in Proposition 4.2 is constructed by a finite combination of these bounded operators acting on $\alpha^1$ and $\alpha^2$, etc, which are smooth on $\overline{U^-}$. For example, $\alpha_1$ in Proposition 4.1, is given by

$$\alpha_1 = \alpha' + (-1)^r \bar{\partial}_K N_{z''} \bar{\partial}_K (K \alpha') + \nu \wedge \bar{\partial}_K \rho + \alpha^2.$$
This implies that for each integer \( m > 0 \),
\[
\|\alpha_1\|_{m(U^c \setminus \mathcal{M})} \leq C(\|\alpha\|_{3m+2(U^c \setminus \mathcal{M})} + \|\nu\|_{m(U^c \setminus \mathcal{M})} + \|\alpha^2\|_{m(U^c \setminus \mathcal{M})}) < \infty,
\]
because each coefficient of \( \alpha', \nu \) and \( \alpha^2 \) is smooth on \( \overline{U^-} \). Note that \( \tilde{\alpha} = \alpha_{q+1} \) is also constructed by a finite combination of these bounded operators acting on some forms which are smooth on \( \overline{U^-} \). By the same reasoning, \( \tilde{\alpha} = \alpha_{q+1} \) satisfies
\[
\|\tilde{\alpha}\|_{s(U^c \setminus \mathcal{M})} < \infty.
\]
for each integer \( s > 0 \). This implies that \( \tilde{\alpha} \in \wedge^{0,r}(\overline{U^-}) \) by Sobolev Lemma. In particular, we have \( \tilde{\alpha} \in \wedge^{0,r}(U^-) \). Since the dimension of \( B \subset \mathcal{M} \) is strictly smaller than \( 2n-1 \)(the real dimension on \( \mathcal{M} \)), it follows that \( (\tilde{\alpha} - \alpha) \wedge \overline{\partial}p = 0 \) on \( \mathcal{M} \cap U \). This completes the proof. \( \square \)

**Proof of Corollary 1.2** Set \( k = \max\{q + 1, n - q\} \). After a holomorphic coordinate change, the Taylor expansion of defining function \( \rho \) for \( \mathcal{M} \) near 0 can be written as in (3.1). We write \( z = (z', z'') \in \mathbb{C}^k \times \mathbb{C}^{n-k} \) as before. Then (3.1) implies that there exist \( \varepsilon > 0 \) and an open subset \( \omega \) of \( M \) such that \( 0 \in \omega \) and \( M_{z'} := \{ z' : \rho(z', z'') = 0 \} \) is strongly pseudoconvex on \( \omega \cap M_{z'} \) as a hypersurface in \( \mathbb{C}^k \) if \( |z''| < \varepsilon \). Then as in the lemma in Section 4 of Bell [2], there is a smooth family of bounded strongly pseudoconvex open sets \( \{D_{z''}\}_{z''|e}\subseteq \) such that \( D_{z''} \subset \mathbb{C}^k \times \{z''\} \) and \( \omega \cap M_{z''} \subset bD_{z''} \) for each \( z'' \). Following the method similar to the proof of Proposition 4.2, we solve \( \overline{\partial} \) equation for \( (0, q) \) forms on the open set \( \cup_{z''|e} D_{z''} \) as follows: Inductively, for each integer \( 1 \leq j \leq q \), we find \( u_j \in \wedge^{0,j}(\cup_{z''|e} D_{z''}) \) such that
\[
\overline{\partial}u_j - \beta = \sum_{J \subset K, M \subset K \subseteq \mathbb{C}^e} H_{JM}d\bar{z}^J \wedge d\bar{z}^M
\]
for some smooth functions \( H_{JM} \) on \( \cup_{z''|e} D_{z''} \). Then as in the proof of Theorem 4.3, \( u = u_q \) satisfies \( \overline{\partial}u = \beta \). The proof is very similar to the previous ones and thus we omit the proof here.

Set \( V^- = \cup_{z''|e} D_{z''} \).

Since we can also solve the extension problem on \( U^- \) by Theorem 1.1, we can solve \( \overline{\partial} \) equation and the extension problem on the common set \( W^- = V^- \cap U^- \). By combining solvability of these two problems, we can solve the \( \partial_b \) problem on \( \mathcal{M} \cap W^- \) as follows: Assume \( \alpha \in \wedge^{0,q} \mathcal{M} \cap W^- \) and \( \partial_b \alpha = 0 \). By Theorem 4.4, there is \( \overline{\partial} \)-closed extension \( \tilde{\alpha} \in \wedge^{0,q}(W^-) \) of \( \alpha \). Then \( u|_{\mathcal{M} \cap W^-} \) satisfies \( \partial_b u|_{\mathcal{M} \cap W^-} = \alpha \) on \( \mathcal{M} \cap W^- \), where \( u \) is the solution of \( \overline{\partial}u = \tilde{\alpha} \) on \( W^- \) and \( u|_{\mathcal{M} \cap W^-} \) is the restriction and projection of \( u \) onto the space \( \mathcal{B}^{0,q}(\mathcal{M} \cap W^-) \). This completes the proof.

**References**


